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THE

ART OF CALCULATION

BY

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GRAPHICS

OR THE

ART OF CALCULATION BY DRAWING LINES

APPLIED ESPECIALLY TO

MECHANICAL ENGINEERING

WITH AN ATLAS OF DIAGRAMS

BY

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PREFACE.

THIS book will not enable the student of Practical Mechanics to dispense with the use of other books treating mechanics in the ordinary manner. If it did, it might be entitled 'Engineering Mechanics developed Graphically;' but in this case it would have been necessary to include in the text a very great deal that would be merely tedious repetition of what is to be found in other well-known books, and the author would have been open to the accusation of 'mere book-making.' On the other hand, if he assumed on the part of the reader a familiar acquaintance with all the higher mathematical developments of mechanical science, and merely showed how to use graphical constructions in applying these results to practice, the book would lose much of its most important and legitimate utility. It is intended to enable those who have a knowledge of elementary mechanics to advance that knowledge to any degree of thoroughness they may find useful, and to apply that knowledge to the every-day problems of engineering science, without the aid of the more complicated portions of algebraic and trigonometrical mathematics or of the differential and integral calculus. Many have no taste or faculty for this latter sort of mathematics; others have not the time needed to keep themselves *au fait* in its use; and, again, it is undeniably true that the solution of many a problem becomes practicable in point of time and ease by the graphic method which would be intolerably tedious and difficult without its aid.

It seems to be one advantage of the graphic method that it requires a more intimate knowledge of the physical natures of the quantities dealt with than does the algebraic method, whose essence is that it proves certain propositions regarding x, y, and z, without the smallest reference to what is meant by x, y, and z. This thorough understanding of fundamentals is particularly insisted on throughout this book; and the reading of Chapter VII., on Vector and Rotor Addition, is intended to assist in the clear comprehension of some mechanical principles which are too much slurred over in ordinary teaching. In Chapter VIII., too, the explanations that are given regarding Moments and the limited sense in which Resultants may be taken as equivalents for their Components, should help in the same direction.

On reference to Chapter II., it will be seen that the whole subject is divided into eight heads. Of these, only two—namely, Arithmetic and Statics—have been dealt with in previous books on Graphics. The introduction of Algebra and Trigonometry treated graphically is a novelty. It is hoped these chapters will be found interesting and useful, but it has not been deemed desirable to devote very much space to them. A good deal also of Chapter III., on Graph-Arithmetic, is believed to be now for the first time published. The Kinematics of Rigid-Bar Mechanisms was first dealt with on the present system in a paper by the author, written in 1884, and read before the Royal Society of Edinburgh in January 1885.

In the application of Graphics to the Statics of engineering structures, the stress-diagrams for linkages containing *beams* and those for linkages built in *three dimensions*—i.e. for solid structures—are both new. The latter especially seems to be an important development of stress-diagram construction, seeing that *all* structures, if rationally investigated, must of necessity be treated as *solid*.

Again, the author has not found in other books on Graphics any directions as to how to deal with structures containing no

joints where only two links meet; with structures on which the loads act at *internal* joints; or with others offering similar difficulties. Nor does he find that the Cyclic Order of procedure in stress-diagram construction is anywhere else explained, much less insisted on, as it ought to be in view of the fact that this is the whole and sole key to the unravelment of difficulties in construction and in interpretation.

Indeterminate Abutment-thrusts is another subject fully explained in this volume, although unfortunately lost sight of in most text-books.

Several shorthand symbols have been introduced, and, it is hoped, will be found acceptable in offices where graphic work is done. The symbols $\begin{smallmatrix} a \\ b \end{smallmatrix}\!> c$; $a \mathrel{\rlap{\,-}{\uparrow}} b$; and $a \mathrel{\rlap{\,-}{\downarrow}} b$ enable one to write out stress-diagram constructions in one-tenth of the space that would be occupied if uncontracted language were used, and the descriptions gain greatly in definiteness and ready comprehensibility as well as in mere conciseness.

The words Locor and Rotor have been adopted and freely used, the latter in a somewhat different sense to that employed by Professor Clifford. The words were very greatly needed, and were only finally chosen after very lengthy consideration and experiment with other possible expressions. The word Pen is another new one, the choice of which involved much deliberation. At one time the author inclined towards the use of 'cell;' but, this having an already established application different from that intended here, the preference was finally given to 'pen.' Again, the Field in which a motion or other vector occurs has proved to be a convenient and expressive phrase. These and the symbols $\#$ and $\#\!\!|$ of vector and locor equality enabled the author, as he hopes, to avoid much diffuseness in, and give greater definiteness to, his diction.

A Glossary of the meanings of all the terms and symbols used that are either new or not quite commonly understood, is given a prominent place at the beginning of the book.

It is hoped that the Index to the Engravings, giving page references to the text for each figure, will be found specially convenient.

It may be useful to insert here a word of warning to ordinary readers and of appeal for consideration to critics, to the effect that it is *not intended* that the text should everywhere be comprehensible without reference to the diagrams, or that the diagrams should be always capable of interpretation without reference to the text.

If the present volume meet with a favourable reception, and no unforeseen obstacle arise, Part II. will be issued at an early date. In this second part it is intended that the subjects dealt with should include 'The Distribution of Stress and Strain;' 'The Strength, Stiffness, and Design of Beams and of Struts;' 'Economy of Weight in Structures;' 'Stresses in Redundant Structures;' 'The Statics and Dynamics of Machines;' 'Frictional Efficiency;' 'Governors;' 'Fly-wheels;' 'Valve Gears;' 'The Practical Thermodynamics of Furnaces, Boilers, and Engines,' including series of curves facilitating the calculation and design of boilers and steam and gas engines; 'The Hydrostatics and Hydrokinetics of Ships and Hydraulic Machines.' Generally speaking, Part II. will deal mainly with *synthetic* problems, and aim more at the *design* of structures and machines than does Part I., which is chiefly *analytic*.

MASON COLLEGE: *October*, 1888.

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GLOSSARY OF SPECIAL TERMS AND SYMBOLS.

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1. *Sectional Tablet*: Square plate of cardboard, porcelain, or slate ruled with small squares, used for Multiplication and Division.
2. *Direction* indicates not only the *lie* of a line, but also the *sense* along that lie. A line has two possible exactly opposite directions, unless it be specified to have one only of these two.
3. *Vector*: a quantity having the two properties, *Magnitude* and *Direction.*
4. *Locor*: A quantity having the three properties, *Magnitude*, *Direction*, and *Position.*
5. *Rotor*: A quantity which, not being a Vector or Locor, is yet capable of accurate graphic representation by a Vector or Locor line; such as a rotation or a force-moment.
6. $=$ means *Quantitative* or *Numerical Equality.*
7. $\mathrel{=\!\!\!\!|\,}$ means *Quantitative* or *Numerical Equality*, combined with *Parallelism* without reference to Direction or Sense.
8. $\mathrel{=\!\!\!\!|\!|\,}$ means *Vector Equality.*
9. $\mathrel{\equiv\!\!\!\!|\,}$ means *Quantitative* or *Numerical Equality* and *Parallelism*, but of *opposite directions.*
10. $\mathrel{=\!\!\!\!|\!|\!|\,}$ means *Locor or Rotor Equality.*
11. $\mathrel{\equiv\!\!\!\!|\!|\!|\,}$ indicates *Locor Oppositcness*; i.e. quantitative equality, identity of position and of opposite directions.
12. *Centre of Volume, Area or Mass*: The point whose position is the average of the positions of all the individual or part volumes, areas, or masses considered, the average being taken with regard to their magnitudes.
13. *Centre-line of Locors*: The line whose position is the average of the positions of all the locors considered, taken with regard to their magnitudes.
14. *Positive sign of Rotation* is used when the rotation is *right-handed* when viewed in the direction counted positive along the axis of rotation. If the motion of a screw through its nut be positive, its rotation is positive if the thread be right-handed, and *vice versâ.*

15. *Field* of a body, or of a group of bodies maintaining their relative positions among themselves invariable, means the *space geometrically attached to the body or bodies*, or the space defined, and measured in, relatively to the body or bodies. The field of a body moves along with the body when this latter is displaced through the field of any other body. The field includes the space both inside and outside the body itself.

16. '*Through*' a field means relatively to the field specified. '*Past*,' '*Over*,' or '*Round*' a body or point means relatively to the body or point specified.

17. *Integral Area, Volume, or Mass Displacement*, means the area, volume, or mass, multiplied by the displacement of the centre of area or of volume, or of mass.

18. *Axis of Rotational Displacement* is the axis round which the actual displacement of a body may be produced by pure rotation without translation.

19. *Axis of Screw Displacement* is the axis round and along which the actual displacement of a body may be produced by pure uniform screw motion.

20. A space completely surrounded, or 'closed,' by lines or surfaces, by bars or plates, is termed a *Pen*.

21. Two contiguous superficial spaces, being called A and B, the dividing line between them is called A B. This line is common to both spaces, and is in this book frequently called the '*joint between the spaces*.'

22. Similarly, the straight line joining two points is called the '*joint between the points*;' the intersection of two lines is called the '*joint of the two lines*;' the surface between two volume-spaces, and common to both, is the '*joint of the volumes*.'

23. Any space P being bounded by, and separated from other spaces by, the straight lines P A, P B, P C, P D, &c., and being open on one side to infinite space, the polygon so formed is called (P) A B C D &c. ∞. When the polygon is closed it is termed a 'pen' and the sign ∞ at the end of the name is omitted, thereby indicating that the space has no side of it open to infinite space; thus (P) A B C D a pen of four sides. In this name the letters must be placed in the order of the sides consecutively, taken either right- or left-handedly; thus the above may be called (P) A D C B. The pen may often for simplicity's sake be called simply P—i.e. by the letter placed in the space—when no ambiguity arises through thus naming it.

24. The joint of two lines A B, B C is called (A B C). When several lines A B, B C, C D, D E have one common joint, i.e. meet in one point, their joint is called (A B C D E). This is the joint of four lines; but if the spaces E and A have a common boundary, it is the joint of five lines, the fifth line being E A. The letters must be placed consecutively in this name in the order in which the spaces are arranged round the joint, either right- or left-handedly; thus the above joint may be also called (E D C B A). The joint of more than three lines may, however,

be more shortly and quite explicitly expressed by naming three only of the spaces, as (A B C).

25. When the joints of the spaces A B C D E, &c., have one common joint (A B C D E), this latter is a point common to all the spaces and may be called the '*point-joint*' *of the spaces*. When the spaces A B C D E completely surround a pen P, the pen (P) A B C D E may be called the '*pen-joint*' *of these spaces*.

26. When a number of lines $p\,a$, $p\,b$, $p\,c$, &c., radiate from one point p, the point p is called the *pole*, and the *pencil of lines* is symbolised collectively by $(p)\ a\,b\,c$, &c.

27. The perpendicular distance of a point p from a straight line $a\,b$ is in this book written $p\,(a\,b)$. Here a and b may be points. If A and B be spaces, the perpendicular distance of the point p from line A B may still be called p (A B). Sometimes, for sake of clearness, this will be written $\overline{p\,(a\,b)}$ or $\overline{p\,(A\,B)}$. The moment of a locor A B round an axis P is written $A\,B \cdot \overline{P\,(A\,B)}$.

28. A couple formed of two equal and opposite locors or rotors A B and C D is written $(A\,B \mathrel{+\!\!\!\!\equiv} C\,D)$.

29. $\circlearrowright$ means *right-handed cyclical order* in stress-diagrams, and $\circlearrowleft$ means *left-handed cyclical order*.

30. The tension stress is counted + ; the compressive stress −.

31. The elongation strain is counted + ; the contraction strain −.

32. Shear stresses and strains, when requiring to be indicated by a sign analogous to + and −, are given the signature L.

33. '${a \atop b}\!> c \,\Big\|\, {P\,Q \atop R\,S}$' means 'draw from the known points a and b parallel to the known directions P Q and R S two lines, and mark their intersection c.' Usually $a\,c$ is *understood* to be drawn parallel to the *similarly named* known line A C, and $b\,c$ to the similarly named line B C; and in this case the complete symbol '${a \atop b}\!> c \,\Big\|\, {A\,C \atop B\,C}$' is contracted to '${a \atop b}\!> c$' simply.

34. '$a \not\downarrow b \parallel P\,Q$' means 'find point b by drawing from the known point a in *plan* a line parallel to the known line P Q, and by projecting vertically downwards on to this line from the known point b in *elevation*.' Usually the line $a\,b$ is *understood* to be drawn parallel to the *similarly named* known line A B, and in this case the complete symbol is contracted to '$a \not\downarrow b$.' The converse process of finding a point in *elevation* by upward projection from its already known plan is indicated by '$a \not\uparrow b$.'

35. '$a\,b = 0$' means that, the length $a\,b$ being found to be zero, the point b is to be marked *coincident* with the *already found* point a.

36. In linkages for which stress-diagrams are drawn, the *thin* lines represent *tie-bars* and *thick* lines represent *struts*.

GRAPHICS.

INTRODUCTION.

THE principles according to which stress-diagrams are drawn out are stated in a very complete and general manner in a paper entitled 'Reciprocal Figures and Stiff Frames,' by Clerk Maxwell, in vol. xxvii. of the *Philosophical Magazine*, 1864; but this paper is quite useless to the ordinary engineer who wishes to know how to apply these principles. The theoretical development of the subject is chiefly due to Cremona, an Italian, and to Culmann, a German. A very great advance in adapting the method to practical requirements, and in simplifying the comprehension and construction of the figures, was made by the introduction of the elegant system of marking the diagrams, commonly known as Bow's Notation, the earlier invention of which is, however, also ascribed to Professor Henrici.

Believing that none of the books published on the subject in the English language is well adapted to the needs of the engineering student who wishes to fit himself for the use of the method in business practice, it has been thought well to attempt to supply this deficiency. To succeed in doing so it is absolutely necessary to illustrate every new statement of general applicability by examples worked out in accurately drawn diagrams; and this will accordingly be done. It is just as essential for the student who wishes to profit by the

study of this book to draw out for himself similar examples as accurately and carefully as he can. There is no subject in which it is more easy to imagine that one understands a problem or theorem, and to find, when one comes to the practical application of it, unexpected difficulties cropping up which bar progress even at the outset. These difficulties are found out only by experience in the art. The way out of such difficulties will as far as possible be pointed out in the course of the present volume.

1. *Advantages and Disadvantages of the Method.*—The great importance and convenience to engineers of this method of calculation have become very generally recognised of late years, but it may be well to make clear what are the grounds upon which, and in what respects, it can claim superiority to the more time-honoured algebraic, trigonometric, and arithmetic methods. The following advantages may be easily recognised:—All the things that engineers deal with have magnitudes, or quantities, which vary 'continuously'—that is, one magnitude may differ from another by an indefinitely small quantity. Designers who never mark on their

Continuous quantity

drawings dimensions involving a fraction of $\frac{1}{16}$ in. may be apt to forget this, but the workman who has to file down the piece to the exact $\frac{1}{16}$ to which it has been designed is in no danger of forgetting it. But numerical representation of continuous increase of magnitude is impossible; to represent a very small difference a very long decimal fraction must be used, and to represent an excessively minute difference is practically impossible by numbers. A line, however, is continuous in magnitude in the same way as all the quantities to be calculated are—the difference between two lines may be indefinitely small. It is, therefore, a very suitable vehicle upon which to mark off the magnitudes to be dealt with.

2. Again, the graphic method brings into strong prominence before the mind the important fact that all 'scales' are

Scales

essentially artificial, and that no two scales for quantities of

different kinds can be alike in any respect. The magnitudes of all sorts of things of different kinds are plotted off along lines. Sometimes an inch length along the line means so many tons, or pounds, or kilogrammes; sometimes so many cubic feet; sometimes a quantity of energy, or of momentum, or a linear velocity; it may even mean so many degrees of angular space, or, perhaps, an angular velocity. No confusion of mind is more common than that of thinking that a quantity of one of these kinds may be in some sense equal to a quantity of another kind, and that both may be represented to the same scale. At first sight the habit of marking off all such quantities as lengths along lines might seem to tend to confirm this confusion of mind. No commoner mistake is made by learners on the threshold of the subject than to ask if they should mark off moments, for example, to the same scale as they have employed for forces, or forces to the same scale as has been used for feet. But they cannot work out thoroughly to the end a single problem in graphic calculation without having this absurd idea very much weakened in its hold on their minds, and it will be driven entirely away before they have completed half a dozen examples. **Scales**

3. Also, the accuracy of the measurements that can be made in this method is in a certain sense proportioned to that actually attainable in quantitative knowledge of facts. It becomes at once apparent to the calculator that absolute accuracy in his results is impossible, and he becomes familiar with the true and important idea that all engineering calculations—and, indeed, all physical calculations—possess only a certain degree of accuracy, and are burdened with a certain amount of unavoidable probable error. Nothing is more silly than the statements we often unfortunately meet with in reports from engineers, who ought to know better, of results calculated to a nicety sometimes a thousand, or even a hundred thousand, times greater than is at all possible in fact. The tensile strength of a material may be stated to five **Degree of approximation**

or six figures, when actually it is not known within 5 per cent. The breaking strength of a girder may be stated by three or four figures, when it really cannot be certainly known within 10 per cent. The horse-power of an engine may be stated to within $\frac{1}{10}$th per cent. when it has been calculated from half a dozen measurements, each of which may be subject to a probable error of from $\frac{1}{2}$ per cent. to 5 per cent. Such mistaken exactitude in calculation is in the first place purely deceptive, and thereby injurious, and in the second place it most surely reveals the calculator in a most ridiculous light to the eyes of those who know better.

Degree of approximation

4. Fourthly, a very large proportion of the quantities engineers have to calculate are quantities having direction, and if these are represented by lines which have corresponding directions as well as magnitudes, the comprehension of the relations among the different elements of the problem becomes much more natural and easy. The word 'graphic' etymologically means 'by line drawing,' and it is in this sense that it is used in the title that has been given by common consent to this art; but in its popular sense of 'peculiarly representative and calculated to assist the imagination' the word 'graphic' may, for the above reason also, be considered particularly appropriate and descriptive. To such directed quantities the general name 'vector' has been given. Examples are:—Distances between two positions, i.e. ordinary dimensions, motions, velocities, momenta, forces.

Vectors

5. These are general considerations in favour of the graphic method, but so far as business utility is concerned they might easily, and in fact are frequently, overbalanced by more practical necessities, such as, for example, rapidity and accuracy. Absolute accuracy, as has been already said, is never attainable, and the pretence of it is in most cases ridiculous; but in many special circumstances it may be essential to obtain as high a degree of accuracy as is by any means possible. In such special circumstances it not infre-

Minute accuracy unattainable

quently happens that the capability of ordinary arithmetic for the attainment of exactitude quite outdoes that of any graphic method. It is, therefore, a mistake to suppose that in every case the graphic method is necessarily the best. With skilful draughtsmanship, however—the special points to be attended to will be mentioned hereafter—an accuracy of $\frac{1}{10}$ per cent. in nearly all cases, and very frequently of $\frac{1}{100}$ per cent., may be obtained.

Minute accuracy unattainable

6. To get at all good results exact drawing is needful, and the endeavour after this exactitude is in itself a good training to every engineer of whatever class.

Training to exactitude

7. Regarding accuracy, also, there is one very important advantage the graphic method possesses—namely, that in a large variety of most important problems the drawing always furnishes a very easy test of its own accuracy, so that it becomes absolutely impossible to make a large mistake without its being discovered.

Automatic test

8. As regards rapidity, graphic calculation is in a great number of cases the most rapid method that can be adopted, provided certain obvious conditions are fulfilled. These are that the calculator has his drawing instruments, pen, paper, pencil, and ink always at hand and in good condition ready for immediate use. If he takes half an hour to get his instruments ready in order to make a small calculation which when these preparations have been completed takes ten minutes more to finish, he must not be surprised if his arithmetical or trigonometrical competitor has beaten him as regards time, and he must not blame the method for his want of success. As paper, drawing-board, and instruments cannot always be ready at hand, this points to a class of calculations for which the graphic method offers no sort of advantage—namely, short, easy calculations. We have heard of the product 4×5 being found with the help of logarithms. We do not admire the method, although we remain fully convinced of the advantages obtained from using logarithms. If one has to

Limiting conditions

make a single multiplication, say $(347\frac{1}{4})^2 \times \cdot 7854$, there is no utility in putting down a piece of paper on a board, sharpening a pencil, and getting out one's scale, in order to find the product. It may be found by ordinary arithmetic, or by reference to a table of areas of circles, in the time that must be spent in sharpening the pencil. If, however, fifty or a hundred multiplications of odd factors have to be performed, it may be worth while to spend the time necessary for the above preparations.

Limiting conditions

9. In these last hundred calculations it is also an important consideration that they involve much less mental fatigue when performed graphically than by any other method. This is one of the greatest, if not the greatest, advantage offered by graphics. In scientific designing, and, indeed, in nearly all engineers' head-work, a large amount of wear and tear of brain power occurs through the need of performing lengthy and laborious calculations that are, to a large extent, mere mental drudgery. Nothing is more fatiguing to the eyes, and through the eyes to the brain, than dealing with interminable arrays of figures. Graphic calculation does away with such fatigue to a very great extent, and in fact a moderate amount of such exercise is felt to be pleasant and enjoyable—merits which few will venture to ascribe to numerical arithmetic. No universal superiority, then, can be ascribed to this art over its rivals. It should be used in such circumstances as bring into prominence its peculiar advantages. Adherence to it under all conditions, and to the entire exclusion of assistance from other modes of reckoning, is to be avoided as dogmatic and irrational, while a denial to it of very great superiority in innumerable practical problems would show equally unreasonable prejudice.

Mental fatigue

CHAPTER I.

INSTRUMENTS.

1. It is only with scrupulously exact drawing that the generally useful degree of accuracy can be obtained in graphic calculations. It is, therefore, of the highest importance that all the instruments used should be true.

2. The *Scale* is of the first importance. It should be engine-divided. No better material can be chosen for the scale than boxwood. Cardboard scales are not sufficiently exact for this class of work. The oval section, useful as it may be for ordinary working drawings of details of machinery and structures, is unsuitable for graphic calculation because of the unsteadiness with which a scale of this form rests on the paper. Long lines have to be marked off, and measured with exactitude; and as one cannot look fairly at both ends of a long line at once, one must be able to hold the scale firmly on the paper, secure against the slightest slipping, **Scales** while one moves from end to end. The flat section shown in Fig. 1 is therefore the best, offering as it does the fullest amount of frictional resistance to slipping over the paper. The flat underside should be plain, without markings of any sort. The two bevel edges may conveniently be divided in inches and in millimètres. It is important that the division should be decimal. Fig. 2 shows a small portion of the length of the scale used in the engineering classes of the Mason Science College. The numbering of the divisions proceeds in one direction only, from left to right. For each edge there are different lines of numbers. The first series on

the inch edge reads inches, or $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$ of an inch, or any submultiple of tenths of an inch. The second series is for reading to $\frac{1}{2}$ in., or $\frac{1}{20}$ths, or $\frac{1}{200}$ths, &c., of an inch. The third is for $\frac{1}{5}$th, or $\frac{1}{50}$ths, or $\frac{1}{500}$ths, &c., of an inch. The millimètre edge has two similar series of numberings. The minute division to half millimètres and to fiftieth-inches is placed at both ends beyond the end line of the scale, and also on each side of the middle line of the scale. The length of the scale should not be less than 20 in. Long lines, such as frequently occur in the diagrams, ought not to be measured in sections. The scale should stretch the whole length of the line, otherwise inaccuracies will accumulate, and much unnecessary loss of time will occur. A scale 30 in. long is often useful.

Scales

3. T-squares are used in the same manner as in ordinary mechanical drawing. The straightness of the edge is of special importance.

T-squares

4. Next to the scale the straight-edges and set-squares are of greatest importance. Two straight-edges—one about 18 in. long, the other 3 ft. 6 in. or 4 ft. long—are convenient. The longer one cannot be dispensed with. Mahogany is a good material, giving a good edge without special edging. If these are edged with ebony, the exactness and smoothness of the edge are greatly improved. It is essential that the straight-edges be kept dead true. They should, therefore, be frequently examined, and trued-up with scrupulous care, if they be found to have bent or warped in the least degree. The same remarks apply to the set-squares—their edges must be kept accurately straight. The right angle of the set-square should be exactly correct. The accuracy of the 60°, 30°, or 45° angle is seldom of any importance, but they should never be used to set off these angles with unless they are known to be correct. Six set-squares should be provided: One 60°, of 12 in. or 14 in. length of side; one 45°, of about 9 in. or 10 in. side; and one 75°, of similar size.

Straight-edges

Set-squares

Three small set-squares about 4 in. to 5 in. side, and with the above angles, viz. 60°, 45°, and 75°, are very useful. Frame set-squares of mahogany edged with ebony are the best, but simple pear-wood set-squares keep their shape and straightness of edge very well if a number of round or oval holes be cut out of the central portion.

Set-squares

5. In graphic calculation lines have constantly to be drawn accurately parallel to each other, and frequently at a considerable distance apart. Parallel rulers are very inaccurate in doing this work. The only accurate and convenient method is to slide a set-square along a straight-edge—which may be an edge of another set-square, if this be of sufficient length. In performing the transfer of the direction across the paper it is important to make the operation as simple as possible. In nearly all cases it is possible to complete the transfer in one sliding operation. It is at first sometimes rather confusing to see in what way the set-square and straight-edge are to be arranged for this purpose, and a beginner has frequent recourse to a tortuous policy of many successive slidings of one set-square over the other. This is seldom necessary, even for the longest transfers. The long straight-edge should be laid on the paper always in the direction of the desired transfer, and as close as convenient to the positions of the line to be drawn and of the already drawn line to which it is to be parallel. Then this last line is found to make with the straight-edge some angle lying between 0° and 90°. The set-square angle is now to be chosen which most closely approximates to this angle between the straight-edge and the line to be drawn—i.e. between the desired direction of transfer and the direction to be transferred. With the two 60° and 45° set-squares there is a choice of six set-square angles. The 60° set-square gives 30°, 60°, and 90°; the 45° set-square gives 45° and 90°. By laying the side of one set-square against the straight-edge, and the side of the other set-square against the hypotenuse of the first set-square, there are

Drawing parallels

obtained the two angles 15° and 75°; because 30° + 45° = 75°, and 45° − 30° = 15°; or 60° − 45° = 15°. These last arrangements are shown in Fig. 3. The single set-square with angles 75° and 15° is, however, very greatly to be preferred to this combination of the two 60° and 45° squares. Putting a set-square edge along the line to be transferred, we may thus place the straight-edge inclined to it by any of the six angles 15°, 30°, 45°, 60°, 75°, and 90°, which angles increase by a difference of 15°. Thus the setting of the straight-edge need never deviate from the exact desired direction of transfer across the paper by an angle of more than $7\frac{1}{2}°$ at the most. This will throw the position of the line to be drawn further from or nearer to the straight-edge than that of the line to which it is to be drawn parallel. Since the sine of $7\frac{1}{2}°$ is about $\frac{1}{8}$, the maximum amount by which it may be thus thrown to one side is about one-eighth the distance of transfer. The excess of the length of the edge of the set-square over that of the line to be drawn will generally cover this deviation of the straight-edge from the exact desired direction of transfer. If it does not do so the line of the edge of the set-square may be extended by laying another set-square against it.

Drawing parallels

Sometimes the direction of the parallel lines nearly coincides with that of the transfer. In this case the straight-edge is simply laid along the given line; then near the position of the desired parallel, a side of a set-square is laid against the straight-edge. Finally, a second set-square is laid against this first, and the first is slid along the second into the required position. When a long line has to be drawn through a given point and parallel to another given long line, and at no great distance from this latter, the best procedure is to take with the pencil compass the distance of the point from the given line; with this as radius strike an arc with centre in the given line as far away as possible from the given point, and to draw a tangent to this arc from the given point.

6. In marking off lengths upon lines in the diagram, it is

necessary, in order to secure exactitude, to prick them off with a needle point. An ordinary fine sewing needle, stuck in a small piece of wood to serve as handle, is a very much better instrument for this purpose than the prickers usually sold with mathematical instruments. If intersections of lines are pricked off with this needle point they become much more sharply defined. The dividers should never be used to take dimensions from the scale with; nor should the points of bows or compasses ever be placed on the scale. **Needle-pricker**

7. The pencil points of the bows and compasses used should be filed flat, with a rounded profile. This rounded profile is best obtained by filing one side of the pencil quite flat, and the other to a rounded conical form. The flat must be perpendicular to the line between the two points of the compasses. If this is not attended to, and if the profile be not well rounded, the compasses or bows will draw circles of slightly different radii according as they lean to the paper on one side or the other. These instruments should have sharp points and stiff, inflexible legs. Two pencils are desirable—one for drawing lines, the other for marking and lettering the diagrams. The latter may be a No. 4 Faber's, and is given a round point. The former may best be a No. 5, and its point should be filed flat and broad, and kept always perfectly sharp. **Pencils**

8. In many of the more complex graphic calculations curves need to be drawn. A set of pear-wood curves assists greatly in this work. Of these, what are called 'French' curves are frequently useful, but 'ship' curves are more generally applicable. A good curve for this kind of work possesses the same character that a 'ship' curve ought to have—namely, the curvature should change gradually and continuously from point to point. It is surprising what a variety of curves may be fairly drawn out with the help of only three or four wooden templates of this sort if these have been skilfully shaped. The characteristic shape of any portion of a curve depends upon the rate at which its curvature varies **Curves**

from point to point. It may be very conveniently stated by specifying the change in the length of the radius of curvature per unit length of arc. Thus the rate of change of radius of curvature might be $\frac{1}{2}$ in. per 1 in. length of arc. If one curve be derived from another by diminishing the lengths of successive small arcs, all in the same ratio, and at the same time diminishing in the same ratio the radii of curvature of these successive small arcs, the two curves will differ simply in the second being drawn to a smaller scale than the first—they will have the same general shape. It would be very convenient if the radii of curvature were marked in figures at the different points of the edges of wooden template-curves. A very useful description of curve is obtained by keeping the above rate of change of radius of curvature per unit length of arc uniform throughout the length of the curve. Take, for instance, the above given example of $\frac{1}{2}$ in. change of radius per 1 in. length of arc, and suppose this rate maintained uniform from point

Curves

to point. Let the radius of curvature at point A be 12 in.; let B C D E F be points in the curve whose distances from A measured along the arc are $\frac{1}{4}$ in., $\frac{1}{2}$ in., $\frac{3}{4}$ in., 1 in., and $1\frac{1}{4}$ in. Call the radius of curvature ρ; for instance, at A, $\rho_a = 12$ in.; then $\rho_b = 12\frac{1}{8}$ in.; $\rho_c = 12\frac{1}{4}$ in.; $\rho_d = 12\frac{3}{8}$ in.; $\rho_e = 12\frac{1}{2}$ in.; $\rho_f = 12\frac{5}{8}$ in., &c. Let the point K be distant from A 24 in. measured along the arc. Here $\rho_k = \rho_a + \frac{24}{2} = 12 + 12 = 24$ in. $= 2\rho_a$. Take now on the curve points L M N O P distant from K by arcs of lengths $\frac{1}{2}$ in., 1 in., $1\frac{1}{2}$ in., 2 in., and $2\frac{1}{2}$ in.; that is, at distances apart double those between B C D, &c. We then have $\rho = \rho_k + \frac{1}{2} \times \frac{1}{2} = 24\frac{1}{4}$ in. $= 2\rho_b$; $\rho_m = 24\frac{1}{2}$ in. $= 2\rho_c$; $\rho_n = 24\frac{3}{4}$ in. $= 2\rho_d$; $\rho_o = 25$ in. $= 2\rho_e$; and $\rho_p = 25\frac{1}{4}$ in. $= 2\rho_f$. Thus this latter portion of the curve differs from the first only in that it is drawn double size. It is clear, therefore, that the different portions of the curve have all the same 'shape' as defined above, and differ only in size or scale; each portion is simply a repetition of any other portion drawn to a larger or smaller scale. With a set of curve-templates designed on this

principle and drawn to a suitably graduated series of 'rates of change of radius of curvature per unit length of arc,' a practically inexhaustible variety of curves may be fairly drawn, whether they be required to be drawn to a large or a small scale. For instance, the series may be the following: Template No. 1, rate of change of radius of curvature $\frac{1}{4}$ in. per 1 in. length of arc; template No. 2, $\frac{1}{2}$ in. per 1 in. length of arc; template No. 3, $\frac{3}{4}$ in. per 1 in. length of arc; template No. 4, 1 in. per 1 in. length of arc; template No. 5, $1\frac{1}{2}$ in. per 1 in. length of arc; template No. 6, 2 in. per 1 in. length of arc; template No. 7, $2\frac{1}{2}$ in. per 1 in. length of arc; template No. 8, 3 in. per 1 in. length of arc; template No. 9, $3\frac{1}{2}$ in. per 1 in. length of arc; template No. 10, 4 in. per 1 in. length of arc, &c. **Curves**

Fig. 4 shows three examples of these curves accurately drawn. A scale of inches is divided off on the edge, and the radius of curvature figured at the chief points. The scale should be marked on both sides for convenience in drawing two-sided symmetrical figures. To ensure perfect symmetry of the second side with the first, it is only necessary to plot off two points, and lay between them the reverse side of exactly the same portion of curve-template as has been used between the corresponding two points of the first side.[1]

9. Curves are frequently drawn with the help of splines of lancewood. If these are well made and in good condition they answer the purpose very well, but if by warping or otherwise the spline has become irregularly bent, then it is very difficult to bend it so as to draw fair curves, the local natural bend or twist in the rod always being reproduced to a greater **Splines**

[1] The equation representing the above kind of curve is—

$$\text{Radius of curvature } \rho = \epsilon^{c_2 - c_1 \tan^{-1}\frac{dy}{ax}}$$

where ϵ^{c_2} is the radius at the point where the curve has its maximum height above the axis of x (c_2, therefore, merely showing the position in which the curve lies on the paper), and c_1 depends on the 'shape' as defined above.

or less extent in its artificially bent condition. It is difficult to preserve splines, even if made of the best wood and perfectly straight and regular at first, so as to avoid comparatively speedy injury by local bending. This is partly produced by the use of the spline, whereby, owing to the imperfect elasticity of the wood, severe permanent set is produced in special places. Carefully made splines of slightly hardened steel of small section would not be subject to this disadvantage, but it would be difficult to harden them as much as would be necessary to avoid permanent set, arising through their continued use, without twisting and bending them in the hardening. Pieces of watch and clock spring of different strengths are often very useful.

Splines

CHAPTER II.

DIVISION OF THE SUBJECT.

GRAPHICS may be divided in correspondence with the ordinarily recognised different methods and subjects of calculation. These are :—(*a*) Arithmetic, (*b*) Algebra, (*c*) Trigonometry, (*d*) Dynamics, (*e*) Tabulation and Analysis of Experimental and Mathematical Results. A few words of explanation regarding each of these sections of the subject may be useful before proceeding to the detailed treatment.

1. *Graph-arithmetic.*—Arithmetic shows how to find increased or decreased quantities when they are altered by given amounts or in given ratios. As the solution of every practical problem involves, and, in fact, to a great extent consists, in a more or less complex series of such operations, the rules of arithmetic are applied continually throughout all graphic constructions. It is thus of great importance to be thoroughly familiar with them, and with the special suitability of each rule for the circumstances to which it is most adapted. **Arithmetic**

2. *Graph-algebra* consists in the solution of equations by drawing straight lines and curves. The usefulness of the method in solving equations, which would be very difficult or impossible to solve by other means, will be illustrated by examples. **Algebra**

3. *Grapho-trigonometry* is the ' solution ' of triangles and other rectilinear figures, that is, the calculation of unmeasured sides, angles, and areas from the sides and angles that have **Trigonometry**

been measured. Applications to surveying measurements will be given.

4. *Grapho-dynamics.*—Dynamics may be considered under three heads :—Kinematics, or the pure geometry of motion ; kinetics, or the laws of motion as dependent on the masses of the bodies moving ; and statics, that special branch of kinetics dealing with cases in which the motions are zero, and the forces in equilibrium. Some simple constructions which apply equally to all three sections of dynamics will first be illustrated.

Dynamics

5. There is great practical convenience in treating statics separately from kinetics ; and since the bulk of the interesting engineering problems to which the graphic method has been applied belongs to statics—e.g. applications to bridge and roofwork—this portion of the subject will be taken before the more difficult problems of the kinetics of moving masses.

Statics

6. The plan of separating kinematics from kinetics has been followed in many modern text-books of high authority. This plan is specially convenient for engineering students because it is frequently desired to study the relative displacements, velocities, and accelerations of velocity of the various parts of a mechanism without references to the masses of the moving parts or to the driving and resisting forces.

Kinematics

A special chapter will therefore be devoted to the Kinematics of Rigid Bar Mechanisms.

7. *Experimental Tabulation.*—The results of a series of experiments—for example, on the relation between the speed of a vessel and the horse-power indicated by its engine, or on the relation between the pressure and temperature of steam—are best made clear by plotting them graphically, i.e. drawing a curve, the rectangular ordinates to which are the values of the quantities whose relation is to be investigated. This assists in the elimination of experimental errors ; it shows the relation found in a very clear manner to the eye, and through it to the mind ; and if a formula is desired to represent the

Co-ordinate tabulation

variation, the curve can be analysed as to its geometrical properties.

Mathematical co-ordinate tabulation

8. *Mathematical Tabulation.*—The best and most accurate formula by which to design some dimension is often complicated and tedious in its application to each special case. This prevents its use in practical life where men are busy and have to economise time. Its use is also prevented by the difficulty of understanding the general meaning or the effect of so complex a rule. These difficulties are entirely done away with if the results of the formula are represented by a curve, and the application of a difficult and cumbrous formula becomes absolutely as easy as that of the most simple. These curves ought to be drawn on square sectional paper, the divisions of which ought usually to be decimal. This plotting out of experimental and mathematical results may be called graphic tabulation.

CHAPTER III.

GRAPH-ARITHMETIC.

1. Simple addition and subtraction can seldom be performed by graphic means with any advantage, so far as ease and rapidity are concerned. Suppose two or more quantities known, and that they are to be added together. The sum can be found by ordinary numerical addition much more easily and quickly than can be completed the process of plotting off the magnitudes to a certain scale along a straight line, each successive length plotted having its left-hand end at the right-hand end of the preceding one, and then reading off to scale the length of the line made up of these separate parts. This is evidently the only possible graphic process of arithmetical addition. If any of the magnitudes are to be subtracted, they are to be measured off in the opposite direction to that of the others—that is, backwards along the line on which these others have already been plotted off. This graphic method of addition is, nevertheless, often convenient as a step in a more lengthy and complex graphic calculation. Suppose that by graphic means we have obtained lines the lengths of which represent to a certain scale certain magnitudes. These magnitudes taken separately may be of no interest, but their sum may be the final object of the calculation, or may be needed in order to continue the calculation to its completion. It would cause more trouble, use more time, and be less accurate to read off each of these parts to scale and add the scaled lengths numerically than to add them graphically by careful use of the dividers, or otherwise, and to read off to scale only the

Addition

Subtraction

resulting sum of the lengths. The scale cannot be read to such minuteness and accuracy as the dividers can be set to, and the sum of the errors in reading the different quantities to scale is therefore always probably greater than that of the errors due to inexact setting of the dividers. Moreover, the error in reading to scale is nearly always in the same direction—either always a little too much or else always slightly too small, the direction of the error depending on the peculiarity of the eyesight of the draughtsman. The error in setting the dividers has not the same invariable character; it is as often positive as negative, and the chances are that numerous errors of this sort will not accumulate, but will more probably neutralise each other to so great an extent that the sum of a large number of errors will be by no means correspondingly large.

Addition and Subtraction

2. Sometimes a quantity can only be found by adding up a very long series of very small parts. The magnitude of each small part in the series may be determined beforehand, but not infrequently it cannot be found until the sum of all the previous parts in the series has been calculated. This kind of addition is called integration. Sometimes, when the law determining the successive values of the small parts is a simple mathematical one, the process of integration is very much simplified by mathematical calculation, as explained in the Integral and Differential Calculus. To attain a moderate approximation to accuracy, the parts require to be taken very small and correspondingly numerous. Thus to integrate by ordinary numerical addition is an immensely tedious operation. The same process, however, may be carried out much more rapidly, and with much less fatigue, by graphic means. In the later parts of this book, when we deal with somewhat complicated constructions, we shall have many illustrations of this graphic integration. As illustrations of the beneficial employment of graphic integration occur only in these somewhat difficult problems, we may pass by the subject for the

Integration

present, promising to return to it when its utility will have become more evident and its interest, therefore, greater.

Integration

The graphic addition and integration of areas will be dealt with under Trigonometry, Chap. V.

3. *Graphic Multiplication.*—The problem is to find the product of two or more known quantities.

Let a and b be the quantities; $x = a\,b$ is to be found. This may be thrown into one of the two forms—

$$\frac{x}{a} = \frac{b}{1}$$

and—

$$\frac{x}{b} = \frac{a}{1}.$$

Multiplication

The graphic construction is to draw two similar triangles, in one of which two sides are made 1 and b (or 1 and a), and in the other of which the two similar sides are a and x (or b and x). This will give us a line x, the length of which to the proper scale represents the product $a\,b$. The pair of triangles may be formed in the two ways represented in Fig. 5.

4. In the first of the Figs. 5, b is associated with 1 in the one triangle and x with a in the other, 1 and a being marked off along the same straight line—or parallels—and x and b lying along the other side of the angle—or parallel sides. The two lines on which $b\,x\,a\,1$ are marked may stand inclined to each other at any angle.

Choice of unit

In the figures the heavy lines indicate the data. The light full lines indicate the lines that require to be actually drawn on the paper. The dotted lines (m_1 and m_2) do not need to be drawn, except at the extremity of m_2, where the intersection has to be marked by drawing a short portion of m_2 across the other line. After marking off $b\ a$ and 1 the edge of a set-square is laid across the end points of b and 1 in the first figure, or of a and 1 in the second figure; then the set-square is slid on another straight-edge until its edge passes through the end point of a in the first figure, or of b in the second figure; finally, in this position the intersection

of m_2 and b produced, giving the extremity of x, is marked with pencil.

5. These two diagrams of Fig. 5 of course give the same length for x; but the first is a better construction than the second. In the first the intersection of m_2 determining the length of x is a more sharply defined point than in the second figure, because the angle between m_2 and x is greater in the first than in the second. This arises from the fact that b is more nearly equal to 1 than is a. There results the rule for the above construction that 1 should be marked off on the same line with that one of the two quantities a and b which differs most from 1. The difference between the two cases is expressed by saying that the triangles in the first are 'better conditioned' than in the second. If in the second the ratio between a and 1 were considerably greater than it is in the drawing, the triangles would be 'ill-conditioned.' As the angle at which two lines cut each other becomes smaller, their intersection becomes less well defined, and the reading of a length to it becomes liable to a greater possible error. This arises in two ways, illustrated in Figs. 6 and 7.

Sharp intersections

6. In Fig. 6 the thickness of the intersecting lines is magnified, each line being shown by a double line. The intersection of the two has really the length marked e on the diagram. The angle of intersection being θ, and the thickness of the line t, it is easily shown that the length of the intersection is—

Possible error

$$e = t\frac{1 + \cos\theta}{\sin\theta},$$

which becomes very rapidly larger as θ becomes smaller. The reading of a length to this intersection must be indefinite within this range e.

7. In Fig. 7 is shown the error in the position of the intersection resulting from drawing one of the lines in a slightly incorrect direction. If the incorrectly drawn line is drawn from a point distant H from the other line, it is easy to prove

Error

that the error e resulting from an angular error ε in the direction is equal to—

$$e = \frac{\varepsilon \mathrm{H}}{\sin^2\theta},$$

Error

which for the same error ε increases still more rapidly as θ decreases than does the error shown in Fig. 6.

8. In Fig. 5, of course, a and b may represent any quantities either of the same kind or of different kinds. For instance, they may both be lengths, and then the desired product is an area. If they are both of the same kind, they can be marked off to the same scale. If they are of different kinds, they must of necessity be represented on different scales. In either case their product cannot be measured to the same scale as either of the factors. What, then, are the relations between the different scales employed in the construction? This is explained at once by observing that a length to represent unity (1) has been marked off. This unit length has been, of course, measured to a certain scale. In the first diagram of Fig. 5 it is set off on the same line as a. Suppose it has been measured to the same scale as has been used for a, then x must be read to the same scale as b, in order that its length read to that scale may numerically equal the product of a and b, the geometrical ratio equation being—

Scale of product

$$\frac{x}{b} = \frac{a}{1}.$$

But if x is to represent, not only the numerical magnitude of $a\,b$, but the real product $a\,b$ itself to a scale of its own, that scale cannot be the same as that of b. The unit of that scale represents unit quantity of the particular kind resulting from the multiplication of a and b. For instance, if a and b are lengths, say in feet, the unit of the x scale is unit area, say 1 square foot; or if a is a length in feet, and b a weight in pounds-weight, the scale of x is one of quantities of work, or of force moments, in foot-pounds. This may be more clearly

understood, perhaps, by considering the equation in the form—

$$x \times 1 = a \times b.$$

This equation may be expressed in words thus: 'x to the scale of a multiplied by 1 to the scale of b equals a multiplied by b,' or 'x to the scale of b multiplied by 1 to the scale of a equals b multiplied by a.' It is plain that the scale of x depends on that to which 1 has been marked off, and that any scale may be adopted for 1, provided a corresponding inverse change is made in the scale to which x is read off. For example, in the third diagram of Fig. 5, 1 is marked off to double the previously used scale. The length in inches obtained for x is just half that obtained in the first two diagrams, but when read to half the scale used for x in these first two diagrams, the same result is obtained as before. To illustrate further: suppose a is a number of pounds weight, say 160 lb., measured to the scale of $\frac{1}{100}$ in. = 1 lb., and b is a number of feet, say 11 ft., measured to the scale of $\frac{1}{10}$ in. = 1ft. Suppose, now, that unity is marked off to the same scale as that of b—that is, $\frac{1}{10}$ in. is marked off as 1. Then x must be read to numerically the same scale as a—that is, to the scale $\frac{1}{100}$ in. = 1 foot-pound. It would be found to be 17·6 in. long, and to this scale would mean 1,760 foot-pounds. Suppose, however, that 1 is marked off to double the b-scale, that is, ·2 in. is taken as unity; then the length obtained for x would be 8·8 in., and this length must be read numerically to half the a-scale, namely, $\frac{1}{200}$ in. = 1 foot-pound; and to this scale it will mean as before 1,760 foot-pounds. Once more, suppose 1 in. taken as unity (1)—that is, ten times as much as represents 1 ft. on the b-scale. Then the length obtained for x will be 1·76 in., and the unit of the x scale must be $\frac{1}{10}$ of the length that represents 1 lb. on the a scale—that is, $\frac{1}{1000}$ in. To this scale x measures as before 1,760 foot-pounds. If 2 in. is taken as unity, the length got for x will

Scale of product

be ·88 in., which, read to the scale $\frac{1}{2000}$ in. = 1 foot-pound, means again 1,760 lb.

In the first two constructions of Fig. 5, 1 = 1 in.; in the third, 1 = 2 in. If $\frac{1}{10}$ in. had been taken as unity, the lines m_1 and m_2 would have been inclined to b and x at a very small angle, and the intersection of m_2 defining the end of x would have been an ill-conditioned one. It is true that a long length can be read with a smaller percentage of error than a short one. If lengths can be read to ·01 in., an error of ·005 in. is ten times more serious in a length of 2 in. than it is in one of 20 in. But the error that may arise from a small inaccuracy in the direction of the line m_2—due either to inexact setting of the set-square to the line m_1, or else occurring in the sliding of the set square from the position m_1 to the position m_2—increases much faster than does the length of x—nearly in the ratio of the square of its length. Also it must be remembered that, if it is possible to read the long x with greater proportionate exactitude, to obtain the long x a short 1 must be used. As the proportionate error in reading x decreases, the proportionate error in marking off 1 increases. It is evident, therefore, that such a length should be adopted for 1 as will make the intersection of m_2 with x as well-conditioned as possible. This result is obtained by adopting for 1 a convenient length as nearly equal either a or b as possible. But it must not be chosen so as to give an awkward scale by which to read x. Thus, if the scales used are parts of inches, 1 may be chosen 10 in., or 5 in., or 2 in., or 1 in., or $\frac{1}{2}$ in. If millimètre scales are used, 200, 100, 50, 20, or 10 mm. may be used as 1.

Scale of product

This rule of arranging the units so as to get well-conditioned triangles cannot always be attended to through long complicated graphic constructions involving series of successive multiplication of a variety of quantities of greatly different magnitudes, because in order to follow it it would be necessary to change the unit and the scales from time to time. This

would lead to hopeless confusion, and in such circumstances it is frequently necessary to work with ill-conditioned triangles. The above considerations are, however, of the greatest possible importance throughout the whole of graphic calculation, and they have, therefore, been presented here very fully. Whenever it leads to no confusion or other inconvenience, the unit should be chosen according to the above explained principle. Whenever it is impracticable to do so, it is well to remember that increased care and exactitude in drawing is necessary whenever intersections at acute angles have to be used. In all cases it is necessary to have a clear conception of the true meanings of the different scales used throughout the diagram, to understand the relations between the scales, and to avoid the confusion of imagining that scales which are essentially different in kind can be in any sense the 'same scale'—that is, for example, that 1 in. = 1 lb., and 1 in. = 1 ft., and 1 in. = 1 foot-pound, and 1 in. = 1 square foot area, are in any sense the same scales, or that they are equal in any way, except that they are to be read numerically in the same manner. While the difference of the scale of x from those of a and of b should be remembered, its relation to these should be clearly comprehended, and the manner in which it is to be deduced from these and from the value taken as 1 should never be lost sight of.

Scale of product

The construction of Fig. 5 can be modified in a great variety of ways according to convenience in special circumstances. The special circumstances result chiefly from the different relative positions on the drawing paper that are found to be occupied by the factors a and b in the course of an extensive graphic calculation. The factors generally result from previous portions of the calculations as lines in certain parts of the drawing. It is not desired to draw them over again in order to perform the multiplication. They are to be used in whatever positions they may happen to have been placed in already. They may be near or distant, parallel,

perpendicular, or oblique to each other. They may both radiate from one point; the extremity of one may lie in some intermediate point of the other; or they may cross each other.

Scale of product

9. The diagrams given in Fig. 8 show the most usually useful variations of the construction. The geometrical proof of each is so simple that it need not be given here. The same dimensions for a and b as are used in Fig. 5 are used throughout, so that the different constructions may be more readily comparable. The circumstances in which each should be used are explained in the attached notes. Throughout, thick lines have been used to indicate the data; thin full lines, the construction lines, that require to be actually drawn; and dotted lines, those that do not need to be pencilled or inked, but only obtained by laying the set square or other edge along them.

Multiplication methods

The construction lines are in each case lettered m_1, m_2, m_3, &c., in the order in which they are drawn. Of course, in the actual constructions none of the letters attached to the diagrams of Fig. 8 are needed, but the numerical value of x as read to scale should be written on the line.

DIVISION.

10. In division we have the converse problem to that of multiplication, and very slight alterations of the constructions given for multiplication serve for division. In multiplication we had $x = a\,b$, or $\frac{x}{b} = \frac{a}{1}$. In division we have $x = \frac{a}{b}$, or $\frac{x}{1} = \frac{a}{b}$. If, therefore, in each construction given for multiplication we make b and 1 change places, the construction is converted to that for the division of a by b. This subject, therefore, requires no further explanation.

Division

Very frequently there occurs a problem in combined multiplication and division, say, to find the quantity $x = \frac{a\,b}{c}$.

This could, of course, be found by first finding a line to represent $a\,b$ by a multiplication construction, and then operating on this line with c by a division construction. But it may be found more directly by a single construction, for it may be written $\frac{x}{b} = \frac{a}{c}$, so that if in the multiplication construction c is marked off in place of 1, the line marked x will be the quantity $\left(\frac{a\,b}{c}\right)$ desired. The diagram as thus modified is shown in Fig. 9, in which a special example is worked out.

Example, Fig. 9.—An engine piston has an area of 343 square inches. The steam pressure on it is 85 lb. per square inch. The whole force is transmitted along the piston-rod, which has an area of 5·23 square inches. What is the stress on the piston-rod section? It is in pounds per square inch $\frac{343 \times 85}{5 \cdot 23}$, and this is calculated graphically, in the diagram (Fig. 9), to be equal to 5,580. The number is easily found, by ordinary arithmetic, to be 5,574·5 lb. per square inch, but the degree of accuracy of the data in a problem of this kind does not justify the answer being stated to a pound per square inch.

Division and Multiplication combined

Similarly, continued multiplication of several quantities can be readily completed in one diagram. Fig. 10 shows an example of the calculation of the weight of water filling a rectangular tank, whose sides are 4·75 ft. and 6·25 ft., and whose depth is 5·15 ft. The weight of water per cubic foot is taken as 62·4 lb., and the product to be found is $62 \cdot 4 \times 6 \cdot 25 \times 5 \cdot 15 \times 4 \cdot 75 = x$. Here we have set off 62·4 to the scale $\frac{1}{10}$ in. = 1 lb. per cubic foot, and in this first multiplication, instead of unity to this scale, viz. $\frac{1}{10}$ in., we have used 100 to this scale, or 10 in. We have set off the sides to the scale 1 in. = 1 ft., and in the second multiplication used 10 to this scale—10 in.—instead of unity; and in the third multiplication have used simply unity—1 in. The scale to which x is

to be read is, therefore, $\frac{1}{100 \times 10} = \frac{1}{1,000}$ in. = 1 lb. The result is obtained correct to four figures. It is read to scale 9540, whereas the exact product is 9540·375. The object of changing the unit from 10 in. to 1 in. is to avoid ill-conditioned intersections. Still better intersections would have resulted from taking 5 in. as the unit for all three multiplications, but this would have involved reading x to the awkward scale of $\frac{1}{50 \times 5 \times 5} = \frac{1}{1,250}$ in. = 1 lb.; x would be read first in inches, then multiplied by 10,000, and finally divided by 8. The combined operation of first calculating what the scale is, and then reading x to that scale, would be altogether too cumbrous.

Continued Multiplication

11. By the help of a piece of finely and accurately divided sectional paper and a straight-edge, multiplication and division can be performed very rapidly and with fair accuracy. The paper should be divided in millimètres or in twentieths of inches, and should be 200 of these small divisions square. A piece of fine cardboard or a flat plate of ground glass or porcelain ruled in this way is better than paper, which shrinks and expands somewhat irregularly as the air becomes dry or damp. The straight-edge may be an edge of a set square, or better, a very fine phosphor-bronze wire held stretched. The readiest constructions for use with this 'sectional tablet' are those shown in Figs. 11 and 12.

Sectional tablet

12. In Fig. 11, for multiplication, at the horizontal distance 1 from zero on the scale of the tablet and at the vertical height a place the straight-edge, letting it also pass through zero. Then at horizontal distance b the vertical height of the line measures the product $a\,b = x$.

Multiplication

In Fig. 12, for division, place the straight-edge from zero to the point at horizontal distance b (the divisor) and vertical height a (the dividend). Then the quotient, $\frac{a}{b} = x$, is the height

Division

of this line at the horizontal distance 1. Here, however, it conduces to accuracy to read off $10\frac{a}{b}$ at the hor zontal distance 10.

13. Similar graphic processes of continued multiplication may be applied to finding the positive and negative integral powers of numerical values, such as a^3, a^6, $\frac{1}{a}$, $\frac{1}{a^4}$, &c. The neatest construction for this purpose is shown in Fig. 13, in which two examples are given of series of positive and negative powers of, first, a value greater than unity; and, second, a value less than unity. Evidently, for numbers much larger than unity, the construction is of no use for high powers, because these increase so rapidly that the lines representing them soon become so long as to extend outside any convenient limits of the paper if the unit chosen be not extremely small. **Integral powers**

Between two cross lines at right angles a series of lines is drawn, starting from a point on one of the cross lines distant unity from their intersection, and each diagonal line of the series being perpendicular to those immediately behind and in front of it. The length a or b, whose powers are wanted, is measured from the centre perpendicular to 1, and the first line of the series is drawn to this point. The construction depends on the similarity of all the triangles in the four angles of the cross. This gives all the integral positive and negative powers.

14. The inverse problem of finding fractional powers, such as $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{7}$, $\frac{3}{8}$, &c., has never been solved by any direct graphic process, although an easy and well-known construction gives the square root. In Fig. 14 is illustrated a graphic or mechanical method by which this problem may be very easily solved with great accuracy and rapidity. **Fractional powers**

Suppose we wish to extract the cube root of a quantity a. From the construction of Fig. 13 it is evident that if we lay off a on the line D in Fig. 14, and from its extremity manage

to draw between D C, C B, and B A three lines at right angles to each other, and so that the last will cut A at unit distance from the centre of the cross, then the distance of the intersection on B of the oblique lines will be equal to $\sqrt[3]{a}$, and the similar distance on C will be $a^{\frac{2}{3}}$. This could be done on ordinary drawing paper only by a tedious process of trial and error. But with the help of some accurately and finely divided square sectional transparent tracing paper, the trials may be completed in a small fraction of a minute. Any line of the sectional tracing paper is laid on the extremity of a, as marked off on the line D. This line is followed by the eye to its intersection with C. This intersection does not lie exactly on any tracing-paper line passing from C to B, but an interpolated intermediate line is easily followed by the eye without actually drawing it in. This is done along to its intersection with B, and from this a similar line followed from B to A.

Fractional powers

This cuts A either further or nearer the centre than unity. If the intersection is further than unity from the centre, the sectional paper is turned round right-handedly a small distance, and the line from the end of a again followed through the three angles of the cross. If it cut A still too far out, the tracing paper is again shifted in the same direction, and the process repeated. By means of a very few of such repeated shiftings the paper is got in the right position, and then the $\frac{1}{3}$rd, $\frac{2}{3}$rds, $\frac{4}{3}$rds, $\frac{5}{3}$rds, &c., powers of a can at once be read off. To facilitate this reading off, the cross lines A B C D are finely graduated, the marks being made clearly visible through the tracing paper. The setting is facilitated by fixing a line of the tracing paper at the extremity of a by a needle-point pricker. In Fig. 14 the tracing paper is set so as to read the fifth root—and all integral powers of the fifth root—of 0·7. It reads $\sqrt[5]{0{\cdot}7} = {\cdot}93$. The accurate value, as found by logarithms, is ·931. The method is thus capable of great accuracy, and is, in fact, of much practical utility. To deal with large numbers in this way is not inconvenient in the same way as

in the graphic construction for integral powers. Thus if, for example, it is desired to find $\sqrt[5]{7,000}$, we first divide 7,000 by 10^5, and deal with the quotient, viz. ·07, instead of with 7,000. Graphically, we find $\sqrt[5]{\cdot 07} = \cdot 585$, and this gives $10 \times \cdot 585 = 5\cdot 85 = \sqrt[5]{7,000}$. The correct number is 5·875.

Fractional powers

In Chaps. VI. and VIII., in treating of Linear Velocities due to Rotations and of Force Moments, graphic processes for the multiplication of a series of factors, taken in pairs, and the simultaneous addition of the products, will be explained.

CHAPTER IV.

GRAPH-ALGEBRA.

1. In ordinary algebra the relation between two quantities varying continuously together is expressed by a letter-symbol equation. The same may be expressed graphically by a line, straight or curved, in one or other of several ways. Generally, one of the following five methods will be convenient.

Five methods

2. (1st.) The two quantities whose continuous variation together is to be expressed are represented by the lengths of the two ordinates to a curve. The scales to which they are so represented are necessarily different if the two quantities are of different kinds; if they be of the same kind, they are conveniently plotted to the same scale. The ordinates are in most cases conveniently taken rectangular, but are not so necessarily. Positive ordinates are measured upwards and to the right hand of the axes; negative ordinates downwards and to the left hand from same axes. For this first graphic mode of algebraic representation, 'square sectional paper' is of very great value, especially if it be accurately ruled. The division of the paper should be decimal. Centimètre or half-inch squares each divided decimally are convenient, but inch squares divided decimally are also useful. Every tenth line being heavy, every fifth line should be of medium thickness. Light blue is the best colour for the sectional lines. If the 'slope,' or tangent of inclination, be readily found by differential calculus, it is of very great assistance in drawing these curves to calculate and make use of this slope.

Linear co-ordinates

3. (2nd.) The two are represented by the length of a line

and the direction of that line, or the angle defining that direction. The line is usually a radius vector from a fixed pole; the angle is measured from a fixed radius vector from the same pole. A right-handed rotation from the fixed zero radius should represent a positive angle; a negative angle should be plotted off left-handedly from this zero direction. The line along which the length is to be plotted being thus determined, outwards from the pole along the side of the angle is to be reckoned positive; the opposite direction, or backwards from the pole along the side of the supplementary angle, should be counted negative. Fig. 15 explains these conventions regarding signs.

The datum or zero radius is P O. The angle θ_A is positive, and with this angle positive lengths are measured from P towards $+A$ and negative lengths towards $-A$. The angle θ_B is negative, and with this angle lengths from P towards $+B$ are positive; those towards $-B$ being negative. Evidently a positive length with the angle $\pi + \theta$ or $180° + \theta$ will lie in the same position as a negative length with the angle θ. Again, no distinction can be made between the angles θ, $\theta + 2\pi$, $\theta + 4\pi$, &c. These ambiguities make this mode of representation inconvenient except when one of the quantities, viz. that represented by the angle, is of a recurrent character; in fact, it is not often suitable except when the one variable is actually an angle. In this latter case, of course, it is eminently advantageous in some respects, but not always even then. Thus, for instance, when the driving moment on an engine crank is to be co-ordinated with the angular position of the crank, this method will give a polar diagram than which nothing could be more intelligible to the eye; but if the work done during any given period of rotation has to be calculated from this diagram, the first method explained above, or the 'square diagram,' is preferable, because in it the area underneath the curve will represent the work. Similar remarks will apply to the representation of the relation between the electro-motive

Polar co-ordinates

force of a dynamo and the angular position of the armature. Circular sectional paper is convenient for this kind of diagram, but is not made as a regular article of sale. The drawing of the tangent of a polar curve at each point plotted is of even greater help in drawing the curve fairly than in the case of curves with rectangular co-ordinates. The 'slope' from the radius, or the tangent of the angle of inclination to the radius, is $r\frac{d\theta}{dr}$, and can often be found simply by the ordinary rules of the differential calculus.

Polar co-ordinates

4. (3rd.) The two variables may be made the two angles formed by two lines drawn from a point in the curve to two fixed foci with the line joining the foci.

Angular co-ordinates

5. (4th.) The two variables may be made the distances of a point in the curve from two fixed foci.

Radial co-ordinates

6. (5th.) The two variables may be two distances from a point in the curve; the one distance being from a fixed focus, the other being the perpendicular distance from a fixed straight line called the 'directrix.'

Focus and directrix

The 3rd and 4th modes of graphic representation are not of general applicability, but are suitable for special problems.

The Straight Line.

7. *Rectangular Equation.*—Using the first diagram with rectangular ordinates, the straight line is the graphic representation of the linear equation

$$y + \mathrm{D}x + \mathrm{F} = 0.$$

$y + \mathrm{D}x + \mathrm{F}$

The simplest method of plotting the line from the given data D and F is to plot y vertically at any two of the three values of *the horizontal ordinate* x, -1, 0 and $+1$. For these points y takes the values $(\mathrm{D}-\mathrm{F})$, $-\mathrm{F}$ and $(-\mathrm{D}-\mathrm{F})$. To obtain accuracy the two points plotted should be as far as possible apart, and, therefore, in general $x = -10$ and $x = +10$, giving $y = 10\,\mathrm{D} - \mathrm{F}$ and $y = -10\,\mathrm{D} - \mathrm{F}$, are the best points to use. If the diagram stretch so far it is still better

to use $x = -100$ and $x = +100$. Of course, careful attention must be paid to the signs of D and of F, either of which may be + or −. If F be −, the line crosses the vertical axis above the origin, and *vice versâ*. If D be −, the line has a right-hand upward slope; if D be +, it has a right-hand downward slope. If the equation be given originally in the form $Dx + Ey + F = 0$, the easiest pair of points to plot out are

$Dx+Ey+F$

$$x = 0 \text{ with } y = -\frac{F}{E}$$

and

$$y = 0 \text{ with } x = -\frac{F}{D}.$$

Fig. 16 is given to illustrate the most simple graphic means of obtaining these points. Very careful attention must be paid in this construction to plotting F, E, and D in the proper directions according to their signs + or −.

8. *Polar Equation.*—

$$r = \frac{C}{\sin(\alpha + \theta)},$$

where C and α are the two constants and r and θ the two variables, is represented by a straight line. For in Fig. 17, where P is the pole and PO the datum direction, and o is the distance at which the line R crosses PO at the inclination $(-\alpha)$, we have

$\frac{C}{\sin(\alpha+\theta)}$

$$\frac{r}{o} = \frac{\sin\alpha}{\sin(\alpha + \theta)}.$$

The constant C, therefore, corresponds to $o \sin\alpha$. To plot the line representing

$$r = \frac{C}{\sin(\alpha + \theta)}$$

from the data C and α, we have, therefore, to measure from the pole along PO, the datum direction, the length

$$\frac{C}{\sin\alpha}\left(\text{when } \theta = o, r = \frac{C}{\sin\alpha}\right),$$

and from the point so obtained draw a line inclined $(-\alpha)$ to P O. If an arc with radius C be first drawn from centre P, and tangent to this arc a line be drawn making the negative or left-hand inclination α with P O, then this line is the desired line R, the radius vector to which from P at the angle θ is r.

The equation may also appear in the form

$$r = o\frac{\cos\alpha^1}{\cos(\alpha^1-\theta)} = \frac{C}{\cos(\alpha^1-\theta)}.$$

$\frac{C}{\cos(\alpha^1-\theta)}$ Here α^1 is the complement of α in the above. From P plot off C along a line, making the right-handed angle α^1 with P O, and from its extremity draw a second line perpendicular to this first. The radius from P to this second line at the + angle θ is r.

9. *Focal Angular Equation.*—Let θ and ϕ be the two variable angles, and c the distance between the foci. Then the equation

$$a\cot\theta + b\cot\phi = c$$

is represented by a straight line (*see* Fig. 18).

To obtain this line plot the foci A and B at a distance c apart. From A and B, and perpendicular to A B, plot b and a. The line is to be drawn through the two points so obtained.

This is evident because when $\theta = 90°$, $\cot\theta = 0$, and, therefore, $\cot\phi = \frac{c}{b}$; and again, when $\phi = 90°$, $\cot\phi = 0$, and, therefore, $\cot\theta = \frac{c}{a}$.

a cot θ + b cot φ To prove that the intersections of lines drawn from A and B at the angles θ and ϕ, connected by the above equation, all lie on a straight line, draw parallels to A B through the tops of a and b. From A draw any line at any angle θ meeting the first parallel at a distance z from the line b. From this intersection draw a line perpendicular to A B to meet the second parallel. The line from B inclined at ϕ to A B must pass through this last intersection, because $z = a\cot\theta$, and, therefore, we must have $b\cot\phi = c - z$. Let P be the inter-

section of the two lines at θ and ϕ from A and B. Let x and y be the rectangular co-ordinates of P referred to b and c as axes. We find

$$x = z\frac{y}{b}$$

and

$$c - x = (c - z)\frac{y}{b}.$$

Eliminating z from these equations, we obtain

$$y = \frac{a - b}{c}x + b,$$

which is the equation of a straight line passing the axis of y at a height b above the origin, and lying at a slope $\frac{a-b}{c}$ from the axis of x. **a cot θ + b cot φ**

If θ^1 and ϕ^1 be the complements of θ and ϕ of the above figure, then the same equation may also take any of the three other forms :—$a \tan \theta^1 + b \tan \phi^1 = c$; or, $a \tan \theta^1 + b \cot \phi = c$; or, $a \cot \theta + b \tan \phi^1 = c$.

Conics.

10. The general equation of the second degree between two variables x and y,

$$\mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{F} = 0,$$

can be represented graphically by one or other of the conics—namely, the ellipse, parabola, or hyperbola, x and y being taken as the co-ordinates, most conveniently at right angles to each other. **Conics**

The conic is—

an ellipse if $4\mathrm{AC} > \mathrm{B}^2$;
a parabola if $4\mathrm{AC} = \mathrm{B}^2$;
and an hyperbola if $4\mathrm{AC} < \mathrm{B}^2$.

Choosing any two axes for the curve to represent the above equation, the *centre* of the conic is to be plotted off with the co-ordinates

$$x = \frac{2\,C\,D - B\,E}{B^2 - 4\,A\,C},$$

$$y = \frac{2\,A\,E - B\,D}{B^2 - 4\,A\,C}.$$

If $B^2 = 4\,A\,C$, the divisor in these fractions becomes zero and the co-ordinates of the centre become infinite; that is, there is no actual centre, the curve being a parabola. If the divisor be positive, the curve is an hyperbola; if negative, an ellipse.

The centre being thus obtained, it is more convenient to proceed with the construction from a new set of rectangular axes through the centre as origin and inclined to the previous axes by the angle $2\tan^{-1}\frac{B}{A - C}$, that is, by an angle θ given by

$$\tan 2\,\theta = \frac{B}{A - C}.$$

If we then take

$$f = \frac{A\,E^2 - B\,D\,E + C\,D^2}{B^2 - 4\,A\,C} + F,$$

Conics and use co-ordinates (called, say, x^1 and y^1) from these new axes through the centre as origin, the curve will be expressed by the equation

$$A\,x'^2 + C\,y'^2 = f.$$

This will be an ellipse if both A and C be positive, and an hyperbola if either A or C be negative; it cannot be a parabola, because in that case f becomes infinite. The semi-major and semi-minor axes of the ellipse are $\sqrt{\frac{f}{A}}$ and $\sqrt{\frac{f}{C}}$.

To construct the ellipse draw round the centre two circles of radii, $\sqrt{\frac{f}{A}}$ and $\sqrt{\frac{f}{C}}$, as in Fig. 19. Draw any radius to cut the two circles, and from its intersections with these two circles draw parallels to the axes. These parallels will meet in a point in the curve. For accurate drawing the points so constructed must be closer together near the major

axis than near the minor axis because of the sharper curvature at these places. The ratio of the two circle-radii being $\sqrt{\frac{f}{A} \times \frac{C}{f}} = \sqrt{\frac{C}{A}}$, the correspondence of the construction with the equation is evident if the equation be written **Ellipse**

$$x'^2 + \left\{ y' \sqrt{\frac{C}{A}} \right\}^2 = \frac{f}{A}.$$

Fig. 20 is introduced here only to show clearly the relation between the ellipse and the hyperbola, and not as a desirable construction for the latter. y' is increased in the ratio $\sqrt{\frac{-C}{A}}$ by means of a pair of circles similar to those used for the ellipse, and this increased length is taken as the height of a right-angled triangle whose horizontal side is the constant $\sqrt{\frac{f}{A}}$. The hypotenuse of this triangle is taken as the horizontal x' for the curve corresponding to y'. The construction is inapplicable to any value of y' greater than $\sqrt{\frac{f}{-C}}$, which would be the maximum value of y' in the ellipse if C had been positive.

Fig. 21 shows an easy construction applicable to the whole range of the hyperbola.

From the centre along axis of y' are plotted $\sqrt{\frac{f}{A}}$ and $\sqrt{A}$; and horizontally from same axis is plotted $\sqrt{-C}$. The oblique line at the slope $\sqrt{\frac{-C}{A}}$ is drawn. This is one of the asymptotes, the other having an equal downward slope. A horizontal line at height $\sqrt{\frac{f}{A}}$ is drawn. From various points in this last line arcs are struck from the centre down to cut the axis of x'. Taking these intersections as the x's, the corresponding y's are obtained by drawing verticals through the upper ends of these arcs down to the asymptote. That **Hyperbola**

the construction expresses the equation is shown by writing the latter

$$x'^2 = \frac{f}{A} + \left\{y' \times \sqrt{\frac{-C}{A}}\right\}^2,$$

the horizontal ordinate to the asymptote being greater than the vertical ditto in the ratio $\sqrt{\frac{-C}{A}}$. The vertices on the axis of x' are distant $\sqrt{\frac{f}{A}}$ right and left hand from the centre.

Parabola

When $B^2 = 4\,A\,C$, the curve is parabolic and has no centre. In this case the equation may be written

$$(\sqrt{A}x + \sqrt{C}\,y)^2 + D\,x + E\,y + F = 0.$$

Draw the two lines representing the equations

$$\sqrt{A}\,x + \sqrt{C}\,y = 0 \qquad \text{(diameter)}$$

and

$$D\,x + E\,y + F = 0 \qquad \text{(tangent).}$$

The *former* line is a diameter of the parabola, and the *latter* is the tangent at the extremity of the diameter. If the perpendicular distance of any point in the curve from the diameter be called p, and the perpendicular distance of same point from the tangent be called t, the curve may be expressed by the equation

$$(A + C)\,p^2 + \sqrt{D^2 + E^2}\,t = 0.$$

To construct the curve the simplest method is to plot out a pair of points calculated from this equation, and then to proceed by the method of tangents. For example, p may be taken ± 1, ± 10, ± 100, or $\pm 1{,}000$, according to the scale of the drawing adopted, the choice depending on the desirability of getting two points fairly far away from the diameter. If $p = \pm 1$ be taken, corresponding to $t = -\frac{A + C}{\sqrt{D^2 + E^2}}$, this furnishes a convenient pair of points, which are to be plotted as in Fig. 22, where the calculation of t for $p = \pm 1$ is performed graphically. The lettering of the diagram will make

its construction plain without further description. The points in the curve for $p = \pm 1$ are marked 9 and 1. The intersection of diameter and tangent is marked O′; that of diameter with chord 9 1 is marked V. The tangents from 9 and 1 intersect in the diameter at a distance from O′ equal to O′V. By plotting this distance to the left hand of O′ along the diameter, these two tangents are obtained. These are then divided similarly and reversely (conveniently, but not necessarily, in equal parts) by points 1, 2, 3, 4, &c. The pairs of points numbered identically are joined by lines which are all tangents to the parabola. The division of the two primary tangents may be continued beyond their intersection, and the construction followed out so as to give any desired length of the parabola.

Parabola

Conic Polar Equations.

11. The equation

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

can evidently be expressed by an ellipse with a and b as semi-major and semi-minor axes in conjunction with a circle of radius either a or b. r is the radial distance from the centre to a point in the ellipse. θ is the angle measured from the major axis to the corresponding radius of the circle from which the ellipse may be derived by projection. It must be noticed that θ is here *not* the angle made by r with the datum direction. The equation represented in this way is, therefore, not a true polar equation in the ordinary sense of the term.

$a^2\cos^2\theta + b^2\sin^2\theta$

The ellipse representing this equation cannot become a circle, because then $a = b$, and the equation reduces to $r^2 = a^2 = b^2$, giving no determination of θ.

12. The equation

$$r^2 = a^2 \cos^2 \theta - b^2 \sin^2 \theta$$

$a \cos \theta + b \sin \theta$

cannot be represented by a conic, although it is easy to construct a polar curve to express it.

The equation

$$r = a \cos \theta + b \sin \theta$$

can be represented by a circle referred to polar co-ordinates. If a and b be plotted at right angles to each other (*see* Fig. 23), the sum of their projections on a line inclined θ to a is

a cos θ + b sin θ

$a\cos\theta + b\sin\theta$. The circle desired is, therefore, to be drawn on the hypotenuse $\sqrt{a^2 + b^2}$ as diameter, and the a extremity of the diameter is taken as the pole. θ being measured from a, r is the length of any line drawn from the pole to the circle.

Conic Focal Angular Equations.

13. The equation

$$a\cot(\theta - \alpha) + b\cot(\phi - \beta) = c,$$

in which θ and ϕ are the two variable angles and a, b, c, α and β are five constants, may be represented by a conic curve. It determines only the *shape*, not the *size* of the conic, because *six* data are required for the complete determination of a conic. Thus our curve may be drawn to any scale, or, in other words, it may be constructed on any base. Take any base A B (*see* Fig. 24), and use A and B as foci, measuring θ at A from A B to the radius from A to a point in the curve, and ϕ at B from B A to the radius from B to the same point

a cot ($\theta - \alpha$) + b cot ($\phi - \beta$)

of the curve. At A and B draw perpendiculars to A B, and plot off from them *outwards*, i.e. away from A B, the angles α and β. At distances a and b from A and B draw lines A′ A′ and B′ B′ perpendicular to these last; these will be inclined α and β to A B.

Along these lines A′ A′ and B′ B′ from the feet of the perpendiculars a and b measure off c. Call the points so obtained 6 on A′ A′ and 0 on B′ B′, and divide the two lengths c c similarly and reversely. In the figure each is divided into six equal parts, numbered from 0 to 6 on A′ A′ *away* from the foot of a, and from 0 to 6 on B′ B′ *towards* the foot of b. Thus 0 2 on B′ B′ equals 0 2 on A′ A′; and 0 2 on A′ A′ plus 26 on B′ B′ equals the constant c. Also, for the point P on the curve corresponding to the points 2 2 on the lines A′ A′

and $B'B'$, the angle $2AB$ being θ and $A'2A$ being, therefore, $(\theta - \alpha)$, evidently 02 on $A'A'$ equals $a \cot(\theta - \alpha)$. Similarly, the angle PBA being ϕ and $B26$ being $(\phi - \beta)$, we have 26 on $B'B'$ equal to $b \cot(\phi - \beta)$. Thus the construction gives $c = a \cot(\theta - \alpha) + b \cot(\phi - \beta)$. Similarly, other points can be obtained at the intersections of $A0$ and $B0$, of $A1$ and $B1$, of $A3$ and $B3$, &c., &c. To complete the curve, the similar and reverse division of $A'A'$ and $B'B'$ must be extended in both directions beyond the part c. In the example given, the curve is an hyperbola, having, of course, two branches. The points on the curve are numbered identically with those on the lines $A'A'$ and $B'B'$, from which they have been obtained. Evidently the curve passes through A and B. As the points taken on $A'A'$ and $B'B'$ recede indefinitely to the right hand, the lines drawn through A and B become in the limit parallel to $A'A'$ and $B'B'$. The corresponding point on the curve is marked ∞. The lines joining it with A and B make the angles α and β with AB; thus the conic is one circumscribing the triangle $AB\infty$. If α or β or both are negative, attention must be paid to this change of sign in plotting them off from the perpendiculars to AB; they should be plotted *towards* AB if they be negative. If both are negative, the curve is always an ellipse.

a cot $(\theta - \alpha)$ + b cot $(\phi - \beta)$

Conic Focal Radial Equations.

14. The equation

$$r_A + r_B = c,$$

as is well known, can be represented by an ellipse, r_A and r_B being the variable distances of a point in the curve from the foci A and B. The distance between the foci may be taken at pleasure, so that there is an infinite number of ellipses which will represent the equation with a given value of c. Similarly, an hyperbola represents

$\begin{cases} r_A + r_B \\ r - r_B \\ r = mp \end{cases}$

$$r_A - r_B = c.$$

If r and p be two variables, the equation

$$r = mp;$$

m, being a constant ratio, can be represented by a conic if r be taken the distance of a point in the curve from the focus and p its perpendicular distance from the directrix. If $m < 1$, the curve is an ellipse. If $m = 1$, it is a parabola; if $m > 1$, it is an hyperbola.

$\left\{\begin{array}{l} r_A + r_B \\ r_A - r_B \\ r = mp \end{array}\right.$

Of course, the same equation between the two variables r and p may be graphically expressed in a simpler manner by a straight line through the origin, taking r and p as rectangular or oblique co-ordinates.

General equations

15. Any equation connecting two variables, no matter how complicated it be, whether it involve high or fractional powers, trigonometrical or exponential functions, is capable of graphic representation in one or other of the five methods mentioned at the beginning of this chapter. As one method is usually much preferable to others as regards ease of construction and applicability to the special problem in hand, the draughtsman will always be repaid for the bestowal of very careful consideration upon the choice of method. The choice should, of course, depend on the use to be made of the curve. A familiar knowledge of miscellaneous curves as explained in books on the more advanced modern geometry will be found of great assistance. Eagle's 'Constructive Geometry' is a useful book of this class. Many curves of special utility in engineering are dealt with in various subsequent chapters of this book.

SOLUTION OF EQUATIONS.

Linear Equations.

16. The solution of the simple equation

$$Dx + F = 0$$

means no more than the finding of the quotient $-\frac{F}{D} = x$.

Simple equations

This can be done by any of the graphic methods of division already explained in the chapter on Graph-Arithmetic. It is only in certain circumstances that any advantage is obtained

by the adoption of the graphic method. But it is instructive to view the graphic solution in another way. Choose rectangular axes and draw the straight line

$$\mathrm{D}x + \mathrm{F} = y.$$

Simple equations

The point where this line cuts the axis of x, i.e. where it makes $y = o$, gives the value of x desired.

17. This immediately leads us to the solution of two simultaneous equations involving two unknown quantities.

$$\begin{cases} \mathrm{D}_1 x + \mathrm{E}_1 y + \mathrm{F}_1 = 0 \\ \mathrm{D}_2 x + \mathrm{E}_2 y + \mathrm{F}_2 = 0. \end{cases}$$

Choose rectangular axes; draw the two lines representing the above two equations. The co-ordinates of the intersection of the two lines give the values of x and y, which are the roots of the equations.

Simple simultaneous equations

18. This method, with the help of a piece of finely and accurately divided sectional paper, is not only rapid (when one has practised it sufficiently to have become familiar with the method of constructive procedure), but is also accurate enough for most practical purposes. For instance, with a piece of paper 200 mm. square—that is, with its square edges each 100 mm. distant from the centre, which is taken as origin—the roots of the equations can be obtained with an accuracy of 3 places of figures. A piece of paper 10 inches square divided in half-inches and twentieths will do equally well, being slightly larger in scale and therefore to most persons more readable. Ordinary sectional paper, however, is not accurate enough in its division, and deforms somewhat irregularly with variations in the hygrometric condition of the air. It is better, therefore, to prepare a piece specially for the solution of equations. Fine hand-made cardboard may be used for this purpose. Finely ground glass, porcelain, or any of the white materials used for memorandum tablets, would be still more suitable, not only because they are more permanent in shape under change of temperature and damp-

Sectional tablet

ness of the air, but chiefly because fine pencil lines may be drawn on them and afterwards rubbed out. If paper or cardboard, which it is desired to preserve permanently for this purpose, be used, pencil lines should not be drawn on it, but the lines should be drawn by stretching across the card fine wires; phosphor-bronze wire being very suitable. Since two wires have to be handled, one must be held stretched by weights after being placed in position. It will facilitate the manipulation if the wire be fastened to the two weights either permanently by soldering, or by being twisted round pegs screwed in the weights for this purpose.

Sectional tablet

If a few simple curves (*see* Fig. 25) be drawn permanently on a sectional tablet of this sort, quadratic and other more difficult equations may be rapidly solved by its help.

Quadratics.

19. The equation

$$\mathrm{A}\,x^2 + \mathrm{D}\,x + \mathrm{F} = 0$$

has the roots $x = -\frac{\mathrm{D}^2}{2\,\mathrm{A}} \pm \sqrt{\frac{\mathrm{D}^2}{4\,\mathrm{A}^2} - \frac{\mathrm{F}}{\mathrm{A}}}$.

Of course each item of this result might be calculated graphically, and the items combined by graphic construction so as to find the two roots. This process is, however, too clumsy and tedious to offer any practical advantages.[1]

Quadratics

Since, however, the equation may be written

$$x^2 = -\frac{\mathrm{D}}{\mathrm{A}}x - \frac{\mathrm{F}}{\mathrm{A}},$$

a rapid graphic solution is possible. Choose axes and draw a curve of squares—i.e. the curve $y = x^2$. Note that this curve, being once drawn carefully in a permanent manner on the

[1] In Eagle's *Constructive Geometry* a solution on this principle is given which looks simple; but it starts with the equation in the reduced form

$$x^2 - 2\,a\,x + b^2 = 0.$$

Really fully half the *work* of the solution of any quadratic consists in finding the quantities a and b to reduce it to this form.

sectional tablet, is sufficient for the solution of *all* quadratic equations.

Now draw the line

$$\mathrm{D}x + \mathrm{A}y + \mathrm{F} = 0,$$

which is the same as

$$y = -\frac{\mathrm{D}}{\mathrm{A}}x - \frac{\mathrm{F}}{\mathrm{A}}.$$

Evidently the two intersections of the line with the curve of squares give the two solutions of the quadratic. The line may be 'formed' without drawing it by simply laying a set-square edge or by stretching a fine wire along its true position. If the line does not cut the curve the two roots are 'impossible' or 'imaginary,' and cannot, therefore, be found, or represented in any way, by graphic means.

Since the curve of squares rises very rapidly, it is more convenient to use the curve $y = \frac{x^2}{10}$, as in Fig. 25. The vertical scale of the diagram being now $\frac{1}{10}$ the horizontal scale, the straight line that must be drawn is **Quadratics**

$$\frac{\mathrm{D}}{10}x + \mathrm{A}y + \frac{\mathrm{F}}{10} = 0,$$

that is, the line has to be drawn through the two points

$$x = 0 \text{ with } y = -\frac{\mathrm{F}}{10\,\mathrm{A}}$$

and $$y = 0 \text{ with } x = -\frac{\mathrm{F}}{\mathrm{D}}.$$

In the example given in Fig. 25 the two roots are $x_1 = +3{\cdot}56$ and $x_2 = -8{\cdot}20$.

20. The solution of simultaneous quadratics involving two unknowns consists in drawing the two conics representing the two equations. Their two intersections give the two root pairs of values of x and y. If the conics do not intersect, the roots are imaginary, and the equations are incapable of graphic solution. **Simultaneous quadratics**

21. The cubic equation

$$Gx^3 + Dx + F = 0$$

is of not infrequent occurrence. It can be solved by drawing the curve

$$y = \frac{1}{100}x^3 \text{ (\textit{see} Fig. 25)}$$

and drawing the straight line

$$\frac{D}{100}x + Gy + \frac{F}{100} = 0.$$

Cubic

Their intersections give the real roots of the equation, there being either three or only one intersection. The curve once drawn will serve for the solution of all such equations. The vertical scale is taken $\frac{1}{100}$ the horizontal scale in order to get the cubes from -10 to $+10$ into the sectional tablet.

22. Similarly, if $f(x)$ be any function of x, and we have to determine the value of x from the equation

$$Af(x) + Dx + F = 0,$$

the graphic solution consists in drawing the curve

$$y = f(x)$$

and the straight line

$$Dx + Ay + F = 0.$$

The intersection or intersections of the line and curve give the real roots of the equation.

General

For instance, an important equation insoluble otherwise except by approximation is

$$A \log x + Dx + F = 0.$$

Draw the curve $y = \log x$, and the above-mentioned straight line; their intersection gives the desired value of x. Similarly, the equations

$$A \sin\theta + D\theta + F = 0,$$

$$A \tan^3\theta + D\theta + F = 0,$$

$$Aa^x + Dx + F = 0, \text{ \&c.},$$

may be dealt with.

23. Generally, if the equation be $f(x) = 0$, it is solved

graphically by drawing the curve $f(x) = y$ down to its intersection with the axis of x. Only that portion need be drawn accurately near the intersection. The proper mode of procedure will be understood from what follows regarding simultaneous equations for two unknown quantities. If the tangent to the curve be easily constructed, it should be used in conjunction with calculation of the points. **f (x)**

24. The function may, however, sometimes be considered as the sum of two functions which are easier to deal with graphically separately than combined. The last equations mentioned above are examples, one part of the whole function being very simply represented by a straight line. Or, again, for instance, the equation—

$$A\sqrt{r^2 - x^2} + Dx + F = 0,$$

in which r is a constant, is most rapidly dealt with by drawing the circle **Special equation**

$$y = \sqrt{r^2 - x^2}$$

and the straight line

$$Ay + Dx + F = 0,$$

the intersections giving the two roots.

25. In such cases, the equation taking the form

$$F(x) + f(x) = 0,$$ **General**

the two curves

$$Y = F(x)$$

and

$$y = -f(x)$$

are drawn, and their intersections give the roots of the equation.

26. Similarly, if we have for the determination of two unknown quantities x and y, the two equations **General**

$$F(x, y) = 0$$

and

$$f(x, y) = 0,$$

the two curves represented by these two equations have to be

drawn, and the values of x and y satisfying the equations measured at their intersections.

27. These various processes must, however, be carried out systematically in order to economise labour. For distinction's sake let the vertical ordinates of $F(xy) = 0$ be called Y, and those of $f(xy) = 0$ be called y.

First, note that it is waste of time to complete the calculation of several points or to draw a considerable stretch of one curve before proceeding with the other. Pairs of points for the two curves should be calculated and plotted for each value of x taken before proceeding to another value of x.

Secondly, note that, although the calculation and drawing of the tangents at the points found assist greatly the good plotting of the curves, there is practically little use of doing this for one of the curves unless it can be easily done also for the other.

Method of procedure

Thirdly, the plotting of one point *and* the tangent at that point is nearly, but not quite, so useful as the plotting of two points without their tangents. Whether tangents should be used or not, therefore, depends upon whether the nature of the curve allows the drawing of a tangent with considerably greater ease and less expenditure of time than would be occupied in the plotting of an additional point.

Fourthly, after plotting a pair of points in the two curves corresponding to a certain x, the next x should be chosen as near the intersection as can be estimated.

Thus the process should be carried out in the following manner:

Choose any value of x, say x_1. Find the corresponding Y_1, and y_1, and plot the two points. If the tangents are to be used, draw them and produce them to their intersection. Choose x_2 in the neighbourhood of this intersection, but rounding off to an easy value to deal with if the finding of Y_2 and y_2 involve any complicated arithmetic. If these latter are to be found entirely by graphic construction, then take x_2 exactly at the

intersection of the first two tangents. Plot now Y_2 and y_2, and draw the two new tangents. Next, through these last two points draw arcs of circles to touch the two pairs of tangents at the two pairs of points 1 and 2. (This can be very quickly done by anyone well practised in circle drawing.) Take x_3 exactly at or close to the intersection of these arcs, and repeat the last process, finding two points 3 and their tangents, and drawing circular arcs through them to touch the tangents at 3 and 2.

The fourth chosen x_4 at the intersection of these arcs will in all probability be found to give Y_4 and y_4 so nearly equal, that a practically exact solution will be obtained by drawing arcs as before through these fourth points. But if not, the process must be repeated once more. In most cases, indeed, it will be found unnecessary to go beyond the third pair of points.

Method of procedure

If, on the other hand, the tangents are not to be used, after finding the first pair of points choose *any* horizontal ordinate, x_2. Find Y_2 and y_2 and plot the points. Draw the chords between the pairs of points 1 and 2, producing them to their intersection. Take x_3 near or exactly at this intersection, and proceeding as before find and plot the pairs of points 3. Now draw arcs of circles through the two sets of three points, 1, 2, 3. Take x_4 at the intersection of these arcs, and plot Y_4 and y_4. Draw the arcs passing through the two sets of three points, 2, 3, 4. Compare the radii of curvature of these latter arcs with the former arcs passing through 1, 2, 3, and estimate a change of radius of curvature from the *averages* through 2, 3, 4, up to those at the *points* 4. In making this estimate consider the average radius through 2, 3, 4, to be the actual radius midway between 2 and 4, and the average through 1, 2, 3, to be the acual radius midway between 1 and 3. Using these estimated radii at the points 4, draw arcs through these points and tangential at these points to the last drawn arcs through 2, 3, 4. The intersection thus found will probably be a very close approxi-

mation to the solution sought for; but if it be not found to be so, the same process must be repeated to a fifth or possibly a sixth pair of points.

In carrying out these processes it must be remembered that each new estimate of x may just as probably as not overstep the mark aimed at—that is, lie on the opposite side of the intersection of the curves to that on which lay the last taken x. Thus the successive x's taken may be alternately greater and less or *vice versâ*.

Method of procedure

Similar general methods of procedure should be followed when the curves chosen to represent the equations are polar or focal. In these cases the drawing of tangents through the points plotted is even more useful than when rectangular co-ordinates are used provided they can be obtained without much labour.

28. Sometimes special graphic methods of trial and error are much more direct and rapid than the above general method. A good example is furnished by the equation

$$\mathrm{A} \sin (2\theta + \alpha) = \mathrm{B} \sin (\theta + \beta).$$

Draw two concentric circles of radii A and B (*see* Fig. 26).

Draw any datum radius P O. From P O set off negatively, i.e. to left hand, the angles $\alpha = \mathrm{O\,P}\,a$ and $\beta = \mathrm{O\,P}\,b$. If either α or β be negative it must be set off to right hand of P O, i.e. in positive direction. Then take any trial point 1 on circle b. With this as centre, and with radius 1 O strike the arc O 1′, cutting circle a in 1′. Then if angle O P 1 were θ, angle O P 1′, would be $2\,\theta$; a P 1′ would be $(2\,\theta + \alpha)$ and b P 1 would be $(\theta + \beta)$. Also the perpendicular distance of 1′ from a P would be $\mathrm{A} \sin (2\,\theta + \alpha)$, and the perpendicular distance of 1 from b P would be $\mathrm{B} \sin (\theta + \beta)$. By trial with the dividers we find the first of these distances much *greater* than the second. Take a second trial point 2 and find the corresponding point 2′ by striking an arc from 2 with radius 2 O. We now find distance of 2′ from a P *less* than distance of 2 from b P.

Assume a third point, 3, and find the corresponding point 3′. We find 3′ *more* distant from *a* P than 3 is from *b* P. The fourth point, 4, chosen makes these distances equal, and, therefore, O P 4 is the angle θ satisfying the equation. In a very short time this angle θ can be found with a degree of accuracy such that the possible error is less than can be measured on the paper, the error, therefore, being in inverse proportion to the scale of the diagram drawn.

CHAPTER V.

GRAPHO-TRIGONOMETRY.

1. We will deal here only with plane trigonometry. We have to make calculations regarding plane figures bounded by straight lines. In doing so we must frequently plot off angles. The instruments called 'protractors' are nearly all of them very rough devices at the best, and are far too untrustworthy for accurate work. The vernier protractor made by Stanley with a silvered divided circle and two opposite arms is a reliable and accurate instrument, but it is costly. The cardboard protractor of 12 in. diameter made by the same maker is also useful, although not so reliable as the other. But as any angle can be set off very easily with ordinary instruments with any desired degree of accuracy, the use of protractors is best wholly avoided.

Protractors

2. The method is the following, and requires the use of a table of chords, such as one finds in Chambers's or in most other mathematical tables. The table found in Molesworth is not sufficient for the plotting of angles taken in surveys, because it gives the chords for every degree only; whereas the angle is read in the field to minutes. First draw from the centre from which the angle is to be plotted a circle of, say, 10 in. radius. On this circle the chord of the desired angle is evidently 10 in. multiplied by the tabulated chord for the given angle. This quantity in inches, taken in the dividers, is set off as the chord of the desired angle. For example, suppose the angle to be plotted is 69° 22′. We find 10 chord 69° 22′ = 11·38, and set this off as the chord on a circle

Plotting angles

of 10 in. radius. Now, from the table we find that 11·39 = 10 chord 69° 27′ and 11·37 = 10 chord 69° 17′. Now, it is possible with ordinary care to set the compasses to $\frac{1}{100}$ in., but much greater accuracy than this is not easily possible. Thus we cannot pretend to set off very accurately as a chord anything between 11·37 and 11·38 or between 11·38 and 11·39. These, as we have seen, correspond on a circle of 10 in. radius to angles differing by 0° 5′. With this size of circle, then, we cannot pretend to plot angles to any greater accuracy than 5′. With very small angles, indeed, the accuracy is increased to $3\frac{1}{2}'$, but with angles larger than the above it is considerably reduced. Thus it is advisable never to plot off by this method angles greater than 45°. The complement of the angle—for instance, 90° − 69° 22′ = 20° 38′—should be plotted off instead. This is also the degree of accuracy obtainable with a circular protractor without vernier, with a divided circle of 20 in. diameter. If a circle of 20 in. radius is used, an accuracy of $2\frac{1}{2}'$ is obtainable in plotting. To make the construction on this large scale requires beam-compasses, and, of course, to maintain this accuracy throughout the diagram it requires to be drawn to a correspondingly large scale.

Plotting angles

3. The 'solution' of any triangle or other plane rectilinear figure is accomplished graphically by plotting it off accurately to scale, and measuring the quantities desired. If it is a length that is to be found, it is measured in the ordinary way; an angle is to be measured by a reversal of the above explained process; and the measurement of areas we will immediately proceed to explain. But it should first be observed that, in plotting off, all the angles required should be set off in the first place upon one and the same circle, and the directions so obtained then transferred by sliding set squares upon straight-edges into the positions required in the drawing. That is, we are not to draw a new 10 in. circle at each new point of the drawing where an angle is to be set off, because such a proceeding would involve the waste of time and labour,

Triangles and recti-linear figures

Datum circle

because it would cover the paper with unnecessary and confusing lines, and because at several of these points it will usually be found that there is not room inside the limits of the paper to use a good-sized circle. The centre of the marking-off circle is conveniently chosen near the middle of the paper. Thus in Fig. 27 let it be required to mark off from the line O A downwards the angle 140°. Instead of doing this directly, we mark off upwards $180° - 140° = 40°$ from A to B, and thus get B O as the desired direction, which can now be transferred to any part of the drawing. Again, let it be required to mark off 70° downwards from O A, the angle A O V^1 being less than 70°. We first mark off 20° to O C^1 from O H $\perp$ O V, then take the chord A C^1 in the dividers and set it off from V to C. We thus obtain C O as the direction wanted.

Height of a spire

We now give two examples of plotting off. In Fig. 28 is shown the calculation of the height of a church spire from theodolite measurements. The theodolite is first placed at a station A. The height of its axis from the ground is measured 4·12 ft. The angle of elevation to top of spire is measured 43° 25′. Another station B is chosen in line with A and top of spire. Distance A B on ground is measured 120 ft. Difference of level of ground at B and A is measured by reading with theodolite at A placed with telescope level on surveyor's levelling staff held at B, 2·34 ft. Theodolite is now placed at B, and height of its axis above ground measured 4·47 ft. The angle of elevation to top of spire is measured 25° 15′. The construction is so simple and so easily understood from the figure that no explanation is needed. The marking off circle is struck from O, the position of the theodolite axis at station B with radius $O H = 5$ in. The line a^1 is drawn parallel to O a, the angle H O a being made 43° 25′. From x the intersection of a^1 with O b is measured X down to the ground line through station A. This is the height required. The distance Y from station A to centre of base of spire may also be directly measured from the diagram. The calculation

of X by ordinary trigonometry involves the solution of the two following equations:

$$(120 + Y) \tan 25^\circ 15' + 4{\cdot}47 - 2{\cdot}34 = Y \tan 43^\circ 25'$$
$$X = Y \tan 43^\circ 25' + 4{\cdot}12.$$

The solution of these equations gives $X = 121{\cdot}207$ and $Y = 123{\cdot}744$.

In Fig. 29 is shown the plotting of a piece of ground surveyed with the theodolite, and through which flows a river, preventing the direct measurement of the two sides C D and A G. All the other five sides are measured in the field, and the external angles between each contiguous pair of sides are also measured. The sum of these external angles measured in semicircles is two more than the number of sides, so that, except as a check on the accuracy of the work, the measurement of one of these angles may be omitted. The sides C D and A G are measured on the plot and stated below the diagram. In plotting off the angles at the left-hand of the diagram, always the difference between the given angle and the nearest greater or less multiple of one right angle is made use of. After plotting D E F G, from G is drawn G *b* parallel to the direction of A B and equal to the known length of B A, namely, 230 ft. Then *b c* is drawn parallel to the direction of, and equal in length to, B C. Then *c* C is drawn ‖ G A, and D C in the known direction to meet *c* C in C. Finally, C B is drawn ‖ *c b* to meet *b* B ‖ A G, and from the intersection B there is drawn B A ‖ to the known direction to meet G A in A. To find C D and A G by ordinary trigonometry involves a considerable amount of tedious preliminary trigonometrical calculation, and the solution of two not very easily formed equations.

Survey across river

4. *Areas.*—The areas of rectilinear figures already plotted can easily be calculated by taking one side as a base and multiplying it by the mean height of the area above that base. The following are amongst the most convenient constructions

Areas

for this purpose. The proofs of the constructions are easily recognised and do not need formal demonstration here.

Fig. 30. *Area of Triangle* $= A =$ base $\times \frac{1}{2}$ Height $= B \times \frac{H}{2}$ or $\frac{A}{B} = \frac{H}{2}$.—To find A, mark off 2 or 20 or 200 or 2,000 along

Triangle

B to scale of figure. Draw m_1 and $m_2 \parallel m_1$. The height of intersection of m_2 with opposite side measures the area; that is, $A \times 1$, or $A \times 10$, or $A \times 100$, or $A \times 1{,}000$ = area of triangle $= \frac{1}{2}$ H B.

5. Fig. 31. *Area of Parallelogram* $= A =$ H B.—Mark off

Parallelograms

1 or 10 or 100, &c., along one side, A = height of intersection of m_1 and m_2, or of m_2 and m_3. Area = A, or 10 A, or 100 A, &c.

6. Fig. 32. *Area of Quadrilateral with Unequal Sides* = A.—The given quadrilateral is indicated by heavy black lines. It is split into two triangles by the diagonal m_1, along which is marked off 2, or 20, or 200, &c. m_2 and m_4 are drawn from end of 2 on m_1; m_3 is drawn $\parallel m_2$ and $m_5 \parallel m_4$ to meet the two sides of the quadrilateral meeting at corner from which 2 was marked off; m_6 is drawn $\parallel m_1$. A is measured perpendicularly

Quadrilaterals

to m_1 or m_6. The construction is equally good whether the diagonal m_1 is $>$ or $<$ 2. The area = A, or 10 A, or 100 A, &c.

Another good construction is the following. The quadrilateral is divided into two triangles by the diagonal m_1, from one end of which a circular arc is struck with radius 2, or 20, or 200, &c. (*see* Fig. 33). A tangent m_3 to this arc is drawn from the other end of m_1, and m_4 and m_5 are drawn $\parallel m_1$. They cut off A on m_3. The radius 2 in this construction must be less than the diagonal m_1. The area = A, or 10 A, or 100 A, &c.

7. Fig. 34. *Areas of Irregular Polygons.*—Reduce to a

Polygons

triangle, and proceed as in Fig. 30, or to a quadrilateral, and proceed as in Fig. 32 or Fig. 33. The given polygon in Fig. 34 is indicated by the heavy lines. The extension of the side

09 is chosen as the base because it is the longest. Draw 12′‖20. Then the triangular area 92′29 equals the quadrilateral area 90129. Similarly draw 23′‖32′ and 34′‖43′. The triangular area 94′49 equals the area of the polygon 9012349. Also draw 87′‖79, and 76′‖67′, and 65′ ‖ 56′, and 54″ ‖ 45′. Then the triangular area 4″ 4′ 4 4″ equals the polygonal area to be calculated—namely, 90123456789. From 4′ mark off 2, or 20 or 2,000, &c., and from the end of this to 4 draw m_1, and draw m_2 from 4″ ‖ to m_1 to intersect 4′4 produced. The length of m_2 from 4″ to this intersection measures the area—that is, the area equals A or 10 A or 100 A, &c.

Fig. 35 shows the method of applying the construction of Fig. 33 to the polygon. It is somewhat shorter and neater than that of Fig. 34. It consists essentially in reducing the area to a triangle whose height is 2, the base of which therefore, of course, measures numerically the area. From any corner 6 describe arc m_1 with radius 2 or 20 or 200, &c. To this arc draw a tangent m_2 from any other corner, such as 0. Use m_2 as a base on which to reduce the polygon, in same manner as in Fig. 34. The polygonal area equals the triangular area 7′, 6′, 6 7′, of which the height is 2. Therefore the area = A or 10 A or 100 A, &c. **Polygons**

8. *Areas with Curvilinear Boundaries.*—These are best calculated by dividing them into parallel strips, each of a width easy to deal with as a multiplier of the mean length or height of the strip. If there is no inconvenience in making them so, all the strips are made of equal width. In this case all the mean heights are first added together and their sum then multiplied by the common width. But trouble and time are often saved by taking the widths unequal. The sharper the curvature of the boundary the narrower ought the strips to be taken. The mean height is obtained in all cases with sufficient accuracy upon the supposition that each small portion of the curve belonging to one strip is parabolic. The approximation shown in Fig. 36 does not imply the supposition that the whole curve **Curved figures**

from end to end is a part of the same parabola. The different small portions are supposed to be parts of different parabolas, such as most nearly coincide with them throughout the small length.

In Fig. 36 the strips are ruled off in the direction of the greatest length of the figure in order to have as few strips as possible, and in order to make the small areas at the ends of the strips between the boundary and the chords as small as possible in proportion to the areas of the strips. The full lines indicate the dividing lines between the strips taken, and the dotted lines are midway between the full lines—i.e. they are the centre lines on which the mean lengths of the strips are to be measured. The first two strips are taken $\frac{2}{10}$ in. wide, the next two $\frac{4}{10}$ in. wide, the next two $\frac{2}{10}$ in. wide, and this leaves a portion of width x X which is an odd fraction of an inch. The mean length of this last portion is $x\,y$, and it is reduced to a strip of width $\frac{2}{10}$ in. $(=\overline{67})$, and of length X Y by the construction shown—i.e. by drawing the line x Y. The mean lengths taken on the dotted lines are not measured to the curve itself, but are taken as the lengths between the chords plus two-thirds the distances between the chords and the curve. This is shown on the small diagram to the right hand. Instead of measuring to n on the curve, or to m on the chord, the mean length is measured to r, $m\,r$ being two-thirds of $m\,n$. The division of $m\,n$ into three parts can be accurately enough performed by the eye, and the points r need not actually be marked as in Figure 36; the length can be read off on applying the scale to the line without the point r being visibly marked on the paper; in fact, the small divisions on the scale assist in the accurate taking of the $\frac{2}{3}$ of the small length $m\,n$. The construction lines need not be drawn in fully as shown in the diagram; it is sufficient to mark their intersections with the curve and with the chords.

Curved figures

Indicator diagrams

9. One of the most useful and interesting examples of the

calculation of areas with curvilinear boundaries is that of steam-engine indicator diagrams. Here, however, it is not the area but the mean height that is really wanted. The mean height might be arrived at by calculating the area and dividing by the extreme horizontal length of the diagram; but this is not the most direct way of proceeding. The ordinary method is to divide this extreme length into an exact number of equal parts, to measure the mean heights of the vertical strips of equal breadth corresponding to these equal parts, to sum up these heights and divide the total by the number of strips. For simplicity's sake the number of parts is usually taken as 10. To facilitate the exact division of the length into 10 equal parts there is used what is called a gridiron parallel ruler—that is, a set of 11 small steel straight-edges all linked together by two cross-bars so that all the 11 edges must always remain parallel and equidistant, but may be placed with larger or smaller spaces between them; in fact, constructed in exactly the same manner as an ordinary parallel ruler, only with 11 instead of 2 bars. It is very important to observe that the heights of the strips have to be measured each on its middle line. Thus, if the corrections for curvature of the top and bottom boundaries of the strips be neglected, as is usual,—that is, if the height be measured simply to these boundaries and not to the two-thirds point between chord and curve—it is really unnecessary to draw the vertical lines dividing the strips, but, instead of these, it is better to draw only the middle lines. The gridiron parallel ruler is first set on the diagram as if to draw the dividing lines between the ten strips. With it so set the widths of the two end strips only are marked with pencil. These are then bisected either with dividers, or, as is usually sufficiently accurate, by the eye simply. Between the centres of the two end strips thus obtained we now set the edges of the parallel ruler so as to enclose *nine* strips between these centres—that is, not using the end one of the series of 11 straight-edges of

Indicator diagrams

the gridiron. We then draw in the 10 centre-lines on which the heights have to be measured.

10. Although to the small scale to which steam-engine indicator diagrams are drawn it is nearly always sufficiently accurate to measure the height of each strip to the curve, yet it must usually happen that the point of cut-off falls somewhere within the boundary of one of the strips, and in this case it would evidently be very inaccurate to measure simply to the upper boundary without correction, especially if the cut-off be sharp. In Fig. 37 are shown two constructions by which the necessary correction may always be made with sufficient accuracy for practical purposes by freehand sketching or by ruling with the help of the gridiron parallel ruler. $c\,d$ is the last part of the 'admission' portion of the diagram, and d

Cut-off is the point of cut-off. $a\,c$ and $b\,f$ are the dividing verticals enclosing one of the strips. It is sufficiently accurate to take $d\,f$ as a straight line, and, therefore, $c\,d\,f$ as a triangle. In construction (A) m_1 is drawn from c to f and $m_2 \parallel m_1$. Then m_3 is drawn from c to the intersection of m_2 with $b\,f$. The height of the strip is to be measured on the centre line to m_3. In construction (B) the height $g\,f$ is bisected in e, and m_1 is drawn through e horizontally. From the intersection of $a\,c$ with m_1 the line m_2 is drawn to g, and the height of the strip is measured to the intersection of m_2 with the vertical drawn through the point of cut-off d. It is evidently unnecessary to actually draw the lines m_1 and m_2 on the paper; it is enough to mark their intersections with the other lines.

11. Indicator diagrams are sometimes measured with the aid of an instrument called a planimeter, which measures areas enclosed by re-entrant curves. The instrument carries

Planimeter a tracing point which is carried round the complete curve, and at the beginning and end of the circuit a difference of readings on a small dial-plate is obtained which measures the area. The most commonly used is Amsler's Planimeter, which, when carefully used, gives very fairly accurate readings. It is

no part of the work of this treatise to describe or demonstrate the action of these instruments. They are, however, very valuable aids in many graphic calculations, and, therefore, they deserve notice here. Wherever an enclosed area bounded by an irregular curve has to be measured, they may be usefully employed. But where, as in the case of indicator diagrams, the real object aimed at is not the area, but the mean height, their use is clumsy, slow, and not to be recommended unless in conjunction with logarithmic calculation. The method employed is, of course, to measure the area by the instrument and divide arithmetically by the length. In dealing with indicator diagrams this division must always be an awkward one, because the length is never any exact simple number or fraction of an inch. If, however, the division be performed with the help of a logarithmic slide rule (of which Fuller's Spiral Slide Rule is the most accurate pattern), and if there be a large number of diagrams to work, the use of the planimeter saves much time and labour.

Planimeter

12. By the above processes the value of a complete enclosed area is found or 'integrated.' It is frequently required to integrate an area *progressively*—that is, to find the *progressive* value of the portion of the area lying behind a straight line cutting through the area, which straight line is gradually moved (or *progresses*) perpendicularly to itself, thus always keeping the same direction or parallel to its original direction.

If the line progress towards the right hand, the portion of the area to be found lies to its left hand, and *vice versâ*. To represent the result of this progressive calculation, usually the best way is to represent the gradually increasing area by the gradually increasing height of a curve measured from a straight base line drawn anywhere perpendicular to the moving line. Such a curve may be called the integral area curve. If y be the height between the boundaries of the curve enclosing the area, measured perpendicularly to the above base line, and corresponding to an abscissa x measured parallelly to the same

Integration

base, the ordinate to the above integral curve corresponds (is rather identical) with the integral of y with respect to x as determined by the integral calculus. Thus the graphic construction of this integral curve, which is about to be explained, is really a method of performing graphically any desired piece of integration. The construction is shown in Fig. 38.

Integration

The irregular curve bounds the closed area to be integrated by strips perpendicular to the line 0 0′. 0 0′ is any line drawn in the required direction (i.e. the direction of progression of the integration). The dotted lines separate off the strips into which the whole has been divided. The portions of the strip-areas lying above the line 0 0′ have been called A B C D, &c., and those lying below the line 0 0′ are called A′ B′ C′ D′. Each line separating these strips is conveniently named by the two letters which name the two strips it separates. Thus the line between strip A and strip B is called A B; that between strip E and strip F is called E F. The vertical centre lines of the strips, on which their mean heights are measured, are called $a' \, a$, $b' \, b$, $c' \, c$, &c. Through any point p in the line 0 0′ produced a perpendicular is drawn, and on this perpendicular all the points $a \; b \; c$, &c., and $a' \; b' \; c'$, &c., are projected. Thus on this perpendicular we have measured upwards and downwards from the point p all the mean heights of the strips A A′, B B′, C C′, &c. On the line p 0 is measured off the length p P, which we may take in the meantime as unity. Then from 0 is drawn 0 1 across the space A and parallel to the line from P to the projected a. Then across B is drawn 12 parallel to P b (projected); then 23 across space C parallel to P c; then 3 4 across space D parallel to P d; and so on with the series of successive lines 4 5, 5 6, 6 7, 7 8, 8 9 across the spaces E F G H I. Similarly, starting again from 0, the lines 0 1′, 1′ 2′, 2′ 3′, 3′ 4′, &c., are drawn across the spaces A′ B′ C′ D′, &c., parallel to P a', P b', P c', P d'. If now fair curves are drawn through the points 0 1 2 3...7 8 9 and 0 1′ 2′ 3′...7′ 8′ 9′, the vertical height between these curves will, to a certain scale to be explained

immediately, measure the integral area to the left hand of the line at which this height is measured.

13. To prove this we notice that the triangle 0 1 1′ is similar to P a a', because 0 1 is parallel to P a, 0 1′ parallel to P a', and 1 1′ parallel to a a'. Therefore the ratio $\frac{1\,1'}{a\,a'}$, is the same as that of the horizontal breadth of the strip A to the distance P p. **Integration. Proof**

Thus, since a a' is the mean height of the strips A and A′ taken together, we have the rectangular area 1 1′ × P p = area of strip (A + A′). That is, 1 1′ measures to a scale numerically equal to the reciprocal of P p the area of the combined strips A and A . To this scale evidently the height of 1 above line 0 0′ measures area A, and the depth of 1′ below 0 0′ measures area A′. Again, suppose the line 1 α drawn parallel to 0 0′. The triangle 1 2 α is similar to P b p, and, since 1 α is the breadth and b p the height of the strip B *lying above* 0 0′, we have the rectangular area 2 α × P p = area B. Thus, since the height of α is the same as that of 1 above 0 0′, we have the height of 2 above 0 0′ measuring to the above-mentioned scale the sum of the areas A and B. Similarly, 3 β measures area C, and 4 γ measures D; so that the height of 3 measures A + B + C, and the height of 4 measures A + B + C + D. Thus the height of each of the points 1 to 9 measures the portion of the area to its left hand lying above 0 0′, and the depth of each of the points 1′ to 9′ measures the portion to its left hand lying below 0 0′.

The point P is called the *pole* of the projection diagram, and P p the *pole distance*. The scale to which those heights measure the corresponding areas is *inversely* as the length chosen for the *pole distance*, P p. As this is a most important point often leading to confusion in engineering problems if not clearly understood, it must be fully explained. **Scale of area curve**

If the diagram be drawn full size and the dimensions of the area be measured in *inches*; and if, further, the pole dis-

tance Pp be taken the natural unit, namely 1 in.; then our equation becomes

Height of integral curve $\times$ Pp = rectangular area of breadth 1 in. and of height equal to height of integral curve = the integral area within the given irregular boundary.

The breadth of this rectangular area being 1 in., its height —namely, that of the integral curve—if read off in inches, will give the number of square inches in the desired area.

But in most practical problems the integral curve would be inconveniently steep, and would soon run out of any ordinary size of paper if Pp were taken so small as 1 in. Suppose Pp be taken 10 in. Then all the heights of the integral curve will become just one-tenth of what they were before. Thus $\frac{1}{10}$ in. = 1 sq. in. is now the scale to which the heights measure the areas. Another way of putting it is, the height of the integral curve to the scale of the given figure (that is, in inches) multiplied by the pole distance to the scale of the projection diagram (namely, 10 in.) equals the desired area in square inches.

Scale of area curve

If Pp were taken 5 in., then the scale would be $\frac{1}{5}$ in. = 1 sq. in.; if it were taken 20 in., the scale would be $\frac{1}{20}$ in. = 1 sq. in.

Now suppose the figure has been measured, say, in feet, and has been plotted off to a scale of, say, $\frac{1}{8}$ in. = 1 ft. It is desired to measure the area in square feet. The projected heights are to the scale $\frac{1}{8}$ = 1 ft. Suppose we find it convenient to take the pole distance equal to 10 in. This to the scale of the projection diagram means 80 ft. Then the height of the integral curve measured in feet to the scale $\frac{1}{8}$ in. = 1 ft. multiplied by 80 ft. equals the area in square feet; or the height of the integral curve measured to the scale $\frac{1}{80 \times 8} = \frac{1}{640}$ in. = 1 sq. ft. equals the area in square feet.

Now suppose that for some reason or other the figure has been plotted with different vertical and horizontal scales. For

example, suppose it to be a plot of a longitudinal section for a railway cutting, in which the horizontal scale is invariably smaller than the vertical scale. Suppose the vertical scale to be $\frac{1}{5}$ in. = 1 ft., and the horizontal scale $\frac{1}{50}$ in. = 1 ft. The projection diagram has the same scale as that of the vertical ordinates—namely, $\frac{1}{5}$ in. = 1 ft. Let 12 inches be the distance found convenient for the pole distance—that is, Pp measures 60 ft. on the scale of the projection diagram. If the horizontal were the same as the vertical scale in our figure, we would now read the heights of the integral area-curve to the scale $\frac{1}{5 \times 60} = \frac{1}{300}$ in. = 1 sq. ft. But since the horizontal scale is only one-tenth the vertical scale, the actual areas are evidently ten times greater than we would find if we supposed the horizontal scale equal to the vertical. Therefore, the scale to which the heights of the integral area-curve must be read is $\frac{1}{10 \times 300} = \frac{1}{3000}$ in. = 1 sq. ft. To take a numerical example, and referring to the strip B and the triangle $1\,2\,\alpha$ of Fig. 38; suppose the mean height $p\,b$ measures $6'' = 30$ ft. to scale $\frac{1}{5}'' = 1$ ft., and breadth $1\,\alpha$ measures $4'' = 200$ ft. to scale $\frac{1}{50}'' = 1$ ft. Then the area is, of course, 6,000 sq. ft. The pole distance being 12 in., we have $\frac{b\,p}{\mathrm{P}p} = \frac{6''}{12''} = \frac{2\,\alpha}{1\,\alpha} = \frac{2\,\alpha}{4''}$, whence $2\,\alpha = \frac{4'' \times 6''}{12''} = 2''$. This height to the scale $\frac{1}{3000}$ in. = 1 sq. ft. means $3{,}000 \times 2 = 6{,}000$ sq. ft. Otherwise expressed: the pole distance is 60 ft. to the scale of the projected mean heights, and the horizontal scale of the area is $\frac{1}{50}'' =$ 1 ft. Then the scale to which areas are to be read off on the integral area-curve is 1 sq. ft. $= \frac{1}{50} \times \frac{1}{60}$ in.

$$= \frac{\text{fraction of inch representing 1 ft. on horizontal scale of area}}{\text{pole distance in feet read to vertical scale of area}}.$$

Scale of area curve

14. One useful application of this area-integration is the storage of water in reservoirs. A curve is plotted, giving the

Water storage

rate of supply from day to day, which, of course, varies with the season. Another is plotted to represent the rate at which water is drawn off. The integral of the area between these two curves is the total excess or deficiency of supply over or below discharge. An example is shown in Fig. 39. The excess curve is alternately positive and negative, and its maxima and minima points occur simultaneously with the crossings of the supply and drain curves. The points at which the excess curve reverses the direction of its curvature (points of inflexion) are simultaneous with those at which the supply and drain curves run parallel to each other, i.e. where these two are at maximum or minimum distance apart.

Water storage

15. Another extremely important application of the integration construction that has just been explained is the calculation of the excess of energy delivered by an engine to the crank-shaft during one portion of the revolution and the deficiency during another. Examples are worked out in subsequent chapters.

Applications

Still another use of the same construction is the calculation of the sheer force on different sections of a beam or girder from the known manner in which the load is distributed over the span. This also is treated of in a future chapter.

16. Referring again to Fig. 38, it is to be noted that we have there determined only ten *points* in each of the two integral curves. It must not be supposed that the diagram is composed of straight lines drawn between these points. To get the intermediate points more accurately than can be done by simply drawing a 'fair curve' through the points already determined, it would only be necessary to divide the area into a larger number of narrower strips, and thus obtain more points. In most practical calculations, however, no great refinement is needed, and the curve may be drawn in with enough accuracy from a few points only, provided the draughtsman is guided in doing so by an intelligent and careful comprehension of the true character of the curve at its various

Integral curve

points. Thus it must be noted that, so long as the *mean height* of the strips is *increasing*, the integral curve *bends upwards*, i.e. its centre of curvature lies above. If the mean height remain constant throughout a certain length, the integral curve becomes a straight line throughout that length : it bends neither upwards nor downwards, but maintains a uniform inclination. If the mean height *decreases*, the curve *bends downwards*. In the ordinary case the mean height increases to a maximum, and then immediately begins to decrease. With this maximum or minimum of the mean height coincides a *reversal* of curvature of the integral area curve. If the mean height becomes *zero* at any point, then at this point the curve becomes *horizontal*. It *slopes upwards* so long as the mean height is *positive*. If the mean height ever become *negative*, the curve slopes *downwards*.

Characteristics of integral curve

17. Although the upper boundary of a strip may be practically a straight line, it does not follow that the integral curve also approximates to a straight line throughout the corresponding stretch. Let (in Fig. 40) A a B b be one of the strips, and let $\alpha\,\beta$ be the corresponding stretch of the integral curve. Since the mean height of the strip increases at a uniform rate from a to b, therefore the *inclination* of the curve $\alpha\,\beta$ increases also at a uniform rate. It is easily recognised that the curve is one of the second degree. It is, in fact, parabolic. Its curvature depends on the steepness of the line $a\,b$. The distance of the curve on the middle line $p\,q$ from the chord $\alpha\,\beta$ may be calculated graphically without trouble if desired. Let the letters $a\,b\,\alpha\,\beta$ be taken to represent the heights of the corresponding points. Then, $t\,v$ being drawn horizontally, $\alpha\,v = \frac{1}{2}\,(\beta - \alpha)$. It can be shown that the distance from chord to curve $t\,y = \frac{1}{2}\,(\beta - \alpha)\,\dfrac{\frac{1}{4}\,(b - a)}{\frac{1}{2}\,(b + a)}$. Now, $a\,r$ being drawn horizontally and $r\,q$ being bisected in s, we have evidently $q\,s = \frac{1}{4}\,(b - a)$ and $p\,q = \frac{1}{2}\,(b + a)$. If, then, we draw the lines $q\,\alpha$ and $p\,v$, and through their intersection draw the line $s\,x$,

Correction

then evidently $\overline{\alpha x} = \overline{\alpha v} \cdot \frac{\overline{qs}}{\overline{pq}} = \frac{1}{2}(\beta - \alpha)\frac{\frac{1}{4}(b-a)}{\frac{1}{2}(b+a)}$, and is, therefore, the distance ty required. It is seldom that this calculation of the versine is needed in practical work, and it must be remembered that the same result—namely, to determine the point y of the curve independently—can be just as well, and probably more quickly, obtained by taking the strip in two parts instead of in one in the general construction for deriving the curve. This latter course is, in fact, the more satisfactory one, because the correction above explained is founded on the assumption that ab is a straight line, which in most cases is only an approximation.

Correction

It must be understood that in all the constructions from Fig. 30 to Fig. 40 the dotted lines represent lines which are not to be actually drawn but only 'formed' by laying the edge of the set-square along them and making the required intersections. The practical draughtsman avoids drawing in as many lines as possible, because unnecessary lines not only dirty the paper when in pencil, but they also make the diagram confused and obscure, and render the useful result less readily perceived by the eye.

Frequently the number of measurements taken, from which the figure has to be plotted, is in excess of the number absolutely required. Such is the case always, for example, in a survey. There is then an *embarras de richesses* of data for the calculation, and the results calculated from different sets of data may not exactly agree. The disagreement shows that small errors have been made in the measurements. Later on we will show how these errors are most equally distributed in the graphic constructions.

CHAPTER VI.

COMBINED MULTIPLICATION AND SUMMATION.—MOMENTS OF PARALLEL VECTORS.

1. In the last chapter the graphic summation of area strips was explained. Each area strip was the product of its mean height and its breadth, and the strips stood *side by side*. Any product of two quantities may be represented by a rectangular area whose sides are the two quantities measured as line magnitudes to suitable scales. A number of such areas can be added together only if they all represent things of the same kind and if the scales for the different areas are the same. **Sum of products** We have now to consider the frequently occurring case in which the areas representing the products *overlap* each other, each of them having one edge lying on the same straight line common to all. In the diagrams necessary for the calculation it is seldom necessary to actually draw in the actual sides inclosing the rectangular area; it is sufficient to draw in their correct directions the two perpendicular dimensions of the rectangle.

2. This problem is of very frequent practical occurrence. The most familiar example is that of the total moment round some axis of several parallel forces; for instance, the total turning moment round the fulcrum of a lever of a number of weights hung on the lever, or the total bending moment on **Parallel forces** any section of a beam due to a number of different loads. The constructions given in this chapter for parallel forces are capable of extension to the case of non-parallel forces, and this development will be dealt with in Chap. VIII.

The general construction applicable to the summation of products of any kind of quantities is shown in Fig. 41. Let the products be A a, B b, C c, &c., the factors A B C being all of one kind, and a b c being all of one kind. From any point X along any straight line lay off the factors a b c to any suitable scale, each factor being measured from X. Then from X draw a line perpendicular to this first. Also draw another perpendicular O C′ in any other position on the paper, and from any point O mark off along this line to any suitable scale the factors A B C successively. These factors are not measured all from one point so as to *overlap*, as the factors a b c are, but are *added* along the line so that O C′ is the sum of the three.

Parallel forces

3. From *any* point p in this line draw a line p P perpendicular to O C′ and, therefore, parallel to X a, and make p P equal to 1, or 10, or 100, &c., to the scale of the factors A B C. Then from a draw the line $a\,o \parallel$ P O and $a\,\alpha \parallel$ P A′. The triangle $o\,a\,\alpha$ is similar to the triangle O P A′, and therefore similar dimensions in the two triangles have all the same ratios; thus the bases O A′ and $o\,\alpha$ have the same ratios as the heights Pp and aX. But O A′ = A and aX = a; therefore, we have P$p \cdot o\,\alpha$ = A $\cdot\, a$. If Pp be taken unity to scale of A, then $o\,\alpha$ to the scale of a will numerically equal the product A $\cdot\, a$. If Pp be taken 10 to scale of A, then ten times $o\,\alpha$ to scale of a will equal A a. Whatever length it be convenient to take for Pp, we have $o\,\alpha$ to scale of a multiplied by Pp to scale of A equal to the product A $\cdot\, a$.

Pole distance

Now draw $b\,b' \parallel o$ X to meet $a\,\alpha$ in b', and draw $b'\,\beta \parallel$ P B′. The two triangles $b'\,\alpha\,\beta$ and P A′ B′ are similar; therefore, the heights b X and Pp have the same ratio as the bases $\alpha\,\beta$ and A′ B′. Thus P$p \cdot \alpha\,\beta$ = B $\cdot\, b$; that is, $\alpha\,\beta$ measures the second product B $\cdot\, b$ to same scale as $o\,\alpha$ measures the first A a. Also we have $o\,\beta$ the sum of the two products.

Again draw $c\,c' \parallel o$ X to meet $b'\beta$ in c' and draw $c'\,\gamma \parallel$ P C′. We have now $\beta\,\gamma$ measuring the product C c to the same scale

as above. Thus we can find the sum of any number of products by this construction.

The scale to which the result is read is inversely proportional to P*p*. The *larger* the factors we have to deal with are, the *larger* must we take P*p* in order to get the result to a sufficiently *small* scale to allow of the sum being represented by a line within the limits of a convenient space on the paper. P is called the 'pole' and P*p* the 'pole distance' of the diagram. The pole distance should be taken a convenient number (to the scale of A B C) by which to multiply the quantity read to the scale of *a b c*. In different practical problems it is convenient to use 1, 5, 10, 50, 100, 500, 1,000, 10,000, and even sometimes 100,000.

The pole may be either to left or right of the vertical O C′. If it is at P′ to the right, the diagram $a\,b'\,c'\,\gamma$ would slope *upwards* intead of downwards, and *a o* would slope downwards; the diagram would be simply reversed on the line *a* X. **Right or left pole**

Suppose all the factors *a b c* to be diminished by the same amount. This is equivalent to shifting X to X′, a distance equal to the diminution of each factor. The rest of the diagram remains unaltered: $o'\gamma'$ now measures the sum of the new products. If the diminution exceed *c*; for example, let it be X X″, then *c* becomes negative and the product C *c* is to be subtracted instead of added. This is still effected without alteration of the diagram. $o''\,\gamma''$ measured to the line $c'\,\gamma$ produced backwards gives the algebraic sum, because $o''\,b''$ measures $(A \cdot \overline{a X''} + B \cdot \overline{b X''})$ and $\overline{b''\,\gamma''}$ measures the negative product $C \,.\, \overline{c X''}$. **Change of axis**

If X be shifted to X‴, to the left hand of the crossing I of the lines *a o* and $c'\gamma$, then the negative products overbalance the positive ones, and the sum $o'''\,\gamma'''$ is to be read as a negative quantity.

4. It is on account of these last peculiarities that this construction is most useful for calculation of the moments of parallel forces or loads on beams or levers. *a* X represents the **Moment**

line of the lever or beam: X the fulcrum or section round which the moments of the forces are to be taken. The *points a b c* are the positions at which these forces act perpendicularly to the line a X—that is, parallelly to the line O C′—and the *quantities a b c* are their leverages around X. A B C marked and added on this line represent the forces. The diagram composed of the two straight lines $\gamma\,\gamma'''$ and and $o\,o'''$ parallel to the two lines P O and P C′ form a complete graphic calculation and record of all the moments this set of forces A B C exert on *any and every* point or section of the lever or beam, because X *may be taken anywhere.* The moment on any section is read off between these two lines $\gamma\,\gamma'''$ and $o\,o'''$ on the vertical drawn through the section. To the right of the intersection I of these two lines the moment may be regarded as positive; to the left of the same crossing it has the opposite sign.

Moment Diagram

5. Immediately under the crossing the moment is zero; that is, round a fulcrum under this crossing the given set of forces would have zero turning moment or would balance so far as moments are concerned. Also it is evident that if a single force equal in magnitude to O C′, the sum of the given forces, were applied to a point of the beam immediately under this same intersection, this single force would produce on all sections the same moment as the given set of forces actually produce. For example, round the section X this single force O C′ applied through I would produce the moment $o\,\gamma$, because the two triangles I $o\,\gamma$ and P O C′ are similar, and, therefore, P$p \cdot o\,\gamma$ equals O C′ multiplied by the horizontal distance of I from $o\,\gamma$.

Axis of zero moment

Any of the forces may be negative, i.e. directed upwards. They are then marked off upwards on the line O C′, and from the beginning to the end of this line still represent the algebraic sum of the forces.

Thus the left-hand diagram gives the 'resultant force' and the right-hand diagram gives the 'resultant moments.' The

first has, therefore, been called the Force Diagram, and the second, the Moment Diagram. It is to be noted that the *pole distance* P *p represents a force.* **Axis of zero moment**

6. All quantities which have, like forces, *directions* and *magnitudes* are called '*vectors*.' These will be explained fully in next chapter. Examples are *motions, velocities, momenta, accelerations of momentum or of velocity*. Vectors whose properties or effects depend upon the definite *position* in space in which they occur, have been called '*rotors*.' In this book, however, they will be called '*locors*.' **Vectors** **Locors**

The *moments* of all locors are calculated in the same way as those of forces, and, therefore, the above construction is applicable to the moments of all kinds of parallel locors. It will be extensively used throughout the rest of this work. In all such diagrams the pole distance represents a *vector*. The extension of the method to vectors that are not parallel will be explained in Chap. VIII.

7. By varying the pole distance the inclination of the two final lines $o\,o''$ and $\gamma\,\gamma''$ is also varied; what may be called their combined 'slope' or the sum of their slopes to the horizontal line a X varies inversely as the number of units taken for the pole distance measured to the vector scale. It follows that, although a shifting of P parallel to O C′ alters the inclination or slope of each of the lines $o\,o'''$ and $\gamma\,\gamma'''$, it does not alter the sum of their slopes. Also any shifting of P either vertically or horizontally does not alter the horizontal distance of the intersection I from any line o X; the point I shifts vertically as P is moved, but it always remains in the same vertical line. This line parallel to the forces in which I lies is called the centre line or resultant line of the locors. **Position of Resultant**

In the case of bending moments on beams and in other similar cases, we wish to determine for each section the resultant moment of all the *forces lying to one side of that section*, excluding the moments of those lying to the other side. The above diagram, Fig. 41, gives this at once. For instance, if the section

to be investigated be at X″, we have A and B acting to the left of this section, and their resultant moment round this section is $o'' b''$: while we have C only to the right hand and its moment (of opposite sign to the former) is measured by $b' \gamma''$. Thus for the resultant moment of all forces to the *left* of any section we measure vertically between $a\,o$ and the diagram outline $a\,b'\,c'\,\gamma$, where $a\,o$ is the final line of the diagram outline going from force to force *towards the left.* For the resultant moment of all forces lying to the *right* hand of any section we measure vertically between $c'\,\gamma$ (or the same produced) and the outline $c'\,b'\,a\,o$, where $c'\,\gamma$ is the last line of the diagram going from force to force *towards the right.*

Partial Moment

8. The lettering of Fig. 41, although almost necessary in explaining and proving the mathematics of the construction, is by no means the best for practical use.

Fig. 42 shows the more convenient method of lettering. The utility of this method increases rapidly with the complication of the figure.

The *spaces* between the force lines are called by the names A B C, &c., and the lines themselves separating these spaces are called by the two letter-names of the spaces they separate; thus, A B, B C, E F, &c., are the lines separating spaces A and B, B and C, E and F, and so on. These lines are of indefinite length. The vectors along these lines are called *a b*, *b c*, *c d*, &c.

Cyclical lettering

They are plotted off successively along a line parallel to the given vector direction. The small letters on this line indicate *points*, and the magnitudes of the vectors are measured between these points. Thus the line between the *points d e* in the vector diagram give the *magnitude* and direction of the vector whose *position* and direction are shown by the line D E on the moment diagram. The pole *p* is placed anywhere at a convenient pole distance. Then parallel to the radii (which need not be actually drawn), from *p* to *a b c d*, &c., are drawn lines *across the corresponding spaces* A B C D, &c.,

these being drawn so as to form a chain of successive links —that is, the two lines across adjacent spaces meeting in the line separating these two spaces. The whole of the space on one side of this diagram outline or chain is named P, the names A B C, &c., being confined to those portions of the spaces between the vector position-lines that lie on the other side from P of the chain. The lines or links of the chain are called P A, P B, P C, &c. Thus, corresponding and parallel lines in the two diagrams are called by the same large and small letters. The series of lines or 'chain' may be referred to as (P) A B C D E F G H, and the pencil of radii from pole *p* may be called (*p*) *a b c d*, &c. This makes a complicated diagram easy to comprehend and easy to read, and it makes it almost difficult to make an error in its construction. The two final lines or links in Fig. 42 are P A and P H, and their intersection I is on the centre line or resultant of the seven locors whose resultant magnitude is *a h*. The resultant moment round any axis is measured between P A and P H.

Cyclical lettering

9. This resultant moment is, therefore, not zero except round axes situated in the line drawn through I in the direction of the locors. This set of locors, therefore, does not balance generally with respect to moment. In order that there should be such balance, there must be zero moment round all possible axes. This evidently can occur only if the two final lines of the chain coincide. If an extra vector *h r* along a line H R were added to the existing set, the line P H would cross the space H to its intersection with this new locor H R, and from this intersection would be drawn a line P R parallel to *p r* in the pole diagram, in which *h r* must first be plotted off. This new line P R can only coincide with P A if the new locor line H R cut P H in the intersection I. In order that the extra locor should produce moment balance, its line, therefore, must pass through I. It is further necessary that P R should have the same direction as P A, which can only be if *r* coincide with *a*, because P R is to be drawn parallel to *p r*.

Balancing locor

Thus the new balancing locor must have the magnitude *h a*—that is, must be equal and opposite in sign to the sum of the other vectors. The sum of the magnitudes of the complete set then becomes zero, and the moment round every possible axis is also zero.

Balancing locor

Thus, if the first given set is to be balanced by a single locor, both its magnitude, its direction, and its position are determined by the condition that it must balance the others.

10. Evidently in the diagram there will be no need for either of the letters R or *r*; *r* coincides with *a* and is not wanted. The letter P will now refer only to the *enclosed* space inside the chain. The chain is now a *closed chain*, its final links meeting in I. The space A extends from line B A to the resultant line through I. The space H extends similarly from line G H to same line through I. Therefore, this last line is now to be called H A. The closed chain or outline of the enclosed space P is called (P) A B C D E F G H.

Closed diagram

11. If, however, the set is to be balanced by two other locors, these may have positions assigned to them independent of the condition of balance, which condition may then be fulfilled by assigning proper magnitudes to the two. A familiar example is that of the two vertical supporting forces at the ends of a beam which keep the loads in balance.

Let R_1 A and H R_2 be the assigned positions of the two balancing locors. Then to complete the diagram there have to be drawn a line P R_1 across the space R_1 from the intersection of P A and R_1 A, and a line P R_2 across the space R_2 from the intersection of P H and H R_2, these lines being parallel to corresponding lines drawn from the pole *p*. But if these two new locors are to balance the set completely, these two lines P R_1 and P R_2 must coincide with each other, and must therefore lie from the first-mentioned intersection of P A and R_1 A to the second-mentioned intersection of P H and H R_2. P R_1 and P R_2 being thus one and the same line, they correspond to one line only in the pole diagram. This line $p\, r_2$ is to be

Two balancing locors

drawn from p parallel to the line P R joining the above two intersections. The point r_2 thus found gives $h\,r_2$ as the magnitude of the balancing locor along H R_2, and $r_2\,a$ as that along R_1 A. Also the chain is once more a *closed chain*, and the letter P should be applied to the space enclosed by it.

It must further be noticed that there is no need of both letters $R_1\,R_2$ and $r_1\,r_2$. The space R may be supposed to stretch from line A R_1 to line H R_2 on the other side of the space P enclosed by the chain to that occupied by A B C, &c. Also, r simply may be substituted for r_2. **Two balancing locors**

The new closed chain is now to be called (P) ABCDEFGHR, the space inside it being P and the whole space outside it being divided by the nine locors into nine spaces, each of which is definitely bounded on three sides by three lines but has no definite outer boundary.

12. This closed chain is still used as a moment diagram; not, of course, of moments of the whole set of nine locors, because the set taken as a whole has zero moment round any and every axis, but of the total moments of all the locors lying on one side only of any line parallel to the locors. The total moment of those lying to one side of any such line round any axis lying in this same line is the length of this line intercepted inside the closed chain, measured to the scale determined by the reciprocal of the pole distance. This moment must be taken (say) positive for the set of locors lying to the left hand and negative for those to the right hand, the sum of the two being zero or the moment of the whole set. If the line pass altogether outside the closed chain, the intercepted length is zero, which means zero moment because we have now the whole set lying to one side. **Rule of signs**

There is an extremely useful convention to notice with regard to the signs of these locor moments. It is the following:—

The pole being taken at any convenient position may stand on the paper either right or left of the line on which

the vectors are marked off successively and added. If the vector line stand to the right of the pole, call the figure a *right-handed* diagram; if to the left, call it left-handed. Now observe the *cyclical order* in which the adding of the vectors leads from letter to letter—i.e. the order in which the letters must be read in the pole diagram to give the true directions of the vectors. Now, taking the sides of the chain surrounding the closed space, P, *in the same cyclical order*, pass from one end to the other of the intercept measuring the moment inside the closed chain, following round that portion of the chain lying to the same side of the intercept as do the locors whose total moment is to be measured. This will give the intercept measured in a definite *direction*, either up or down; up for the locors lying on one side and down for those lying on the other side of the intercept. The rule is now that an *upward intercept* means a positive or *right-handed moment if the diagram be right-handed*, and if it be a *left-handed diagram* an *upward* intercept means a negative or *left-handed moment. Vice versâ*, a downward intercept means a left-handed moment on a right-handed diagram, and a right-handed moment on a left-handed diagram. This rule is of universal application, and may save considerable confusion of mind if the diagram be a complicated one with the sides of the chain crossing each other several times.

Rule of signs

A precisely analogous convention will be shown hereafter to be of the greatest utility in stress diagrams for complicated frameworks.

In this construction we have assumed the *positions and directions* of the two balancing locors as known, and that these directions are *parallel* to the other known locors.

Abutment thrusts

13. In the cases of such structures as roofs and bridges resting on walls, piers, or abutments, it is seldom known in which directions the supporting forces are exerted; the 'thrusts' or horizontal components of these forces are unknown. All that is known is that they are exerted through

certain comparatively small surfaces which we may assume in our diagram as known *points.* If this condition is all that is known regarding the action of these forces, it leaves the problem of calculating their amount and direction to a certain extent insoluble. But only to a certain extent. Although they cannot be completely determined, these forces have certain determinable relations which tell the engineer a great deal about them that is useful. Later we will find how these relations enable the forces to be completely determined as soon as certain *coefficients of pliability* of the abutments and of the structure are known. Meanwhile the graphic calculation of these relations is the following in Fig. 43.

The diagram shows in heavy lines the outline of a girder which is taken as sloping, that is, with supports not at the same level, for the sake of greater generality, the method being equally applicable and easy whether these supports are on the same level or not. The loads applied are supposed to be four only, their lines being A B, B C, C D, D E, and their magnitudes, *a b*, *b c*, *c d*, *d e*. The supporting points or foundations are marked F_1 and F_2. Through these are drawn lines (dotted in the figure) parallel to the loads. A pole, *p*, is chosen at a convenient distance and the chain (P) A B C D E drawn, the sides being drawn parallel to the radii from *p*. The intersections of P A and P E with the lines through F_1 and F_2 are joined, and parallel to the line (P R) joining them is drawn *p r*. Then we know that if the forces at F_1 and F_2 were parallel, their magnitudes would be *e r* at F_2 and *r a* at F_1. But if they are not parallel, then each of them may be resolved into two components, one parallel to the loads and the other along the line F_1 F_2. These latter components parallel to F_1 F_2 may again be resolved each into a component parallel to the load and one perpendicular to the load. These perpendicular components must have equal and opposite magnitudes in order that the whole system may balance, because all the other forces in the system are at right

Abutment thrusts

angles to them. Since the two primary components along $F_1 F_2$ and exerted at F_1 and F_2 are in the same direction, and since their secondary horizontal components have the same magnitude, therefore these two primary components have also equal and opposite magnitudes. Thus, each of the supporting forces being resolved into a component parallel to the loads and a component parallel to $F_1 F_2$, which we may call the 'abutment line,' we find that the latter components parallel to the 'abutment line' are equal and opposite. Since they also act along the same straight line, their moments round any and every axis exactly neutralize each other. Therefore, the rest of the system, exclusive of these two, must independently balance with respect to moments; that is, the two remaining components parallel to the loads of the forces at F_2 and F_1 must be $e\,r$ and $r\,a$, the same as if the whole supporting forces were vertical.

Abutment thrusts

Through r draw the line $r\rho \parallel F_1 F_2$, the abutment line. We now see that the force at F_2 is represented to scale by a line from e to some undetermined point in the line $r\rho$, for it is compounded of $e\,r$ and a force parallel to $r\rho$; and that the force at F_1 is similarly represented by a line from the *same point* in $r\rho$ to a. The indeterminateness of the problem is now expressed by the indefiniteness of the point ρ at which the two forces $e\rho$ and ρa meet in the force diagram.

If any one other particular concerning either of the two supporting forces be known, then the problem becomes capable of solution. For instance, if it be known that the support at F_1 be capable of exerting only a horizontal force, then the point ρ can be at once found by drawing from a a horizontal line to intersect $r\rho$. If either force be known to be wholly vertical, then the point ρ will coincide with r and the horizontal thrusts be zero. If the direction of the force at F_2 be known, a line can be drawn from e parallel to that direction to intersect $r\rho$. Thus, in the special example shown, if the force at F_2 were horizontal, or had a less inclination to the horizontal

than $r\,\rho$, the intersection would fall to the right hand of $a\,e$ and the force would be a *tension*. It will, however, be afterwards shown that, if the direction of one of these forces be known, their magnitudes can be found by a slightly more direct and expeditious construction than the one here explained. As a further example of the utility of the above construction, suppose that the horizontal component of either of the forces exerted at F_1 and F_2 be known. This horizontal component is to be plotted off in the proper direction to the right or left hand of the line $a\,e$, and at the distance so obtained a vertical line is to be drawn to intersect $r\,\rho$. This intersection will then be the true position of ρ.

Abutment thrusts

CHAPTER VII.

VECTOR SUMMATION.

1. There is a large series of different kinds of things many of the useful calculations regarding all of which are made by exactly the same methods because the various members of this group have two characteristics in common. **Vectors** They all have *magnitude* or *quantity* and they all have *direction*. This circumstance has led to its being convenient to give a distinctive name to this class—namely, *vector*. Everything that has both *direction* and *quantity* is a vector, whatever be the other qualities it may possess. A vector may, therefore, be shortly and completely defined as a *directed quantity*.

2. Here the word *direction* is used in its strictest sense. A given line in space does not necessarily have a definite direction, or rather it has *two* possible directions exactly opposite to each other. It is not a vector unless it is specified to have one and one only of these two opposite directions—that is, to be directed *from* one specified end of the line *towards* the other end. When two lines have the same *lie* **Direction** in space, if they have opposite directions or if one or both have no specified direction, their coincidence in *lie* is expressed by the word *parallelism*; they are said to be *parallel*. The word 'direction' has often been used to mean the same as 'lie' in the last sentence, and to avoid ambiguity the word 'sense' has been employed to convey the stricter meaning ascribed above to 'direction.' But, although we have no other word meaning exactly what has been expressed above by 'lie,' there seems to be no practical need of using 'sense.' To say

that two lines have the same 'lie' would, apart from the context, be awkward and ambiguous, but the same meaning can be exactly expressed by saying that they 'are parallel.' In this book the word direction will be used to indicate one only of the two possible opposite directions along a straight line.

Direction

Different kinds of vectors have, of course, other properties besides direction and magnitude which distinguish them one from another. But when dealing with them as vectors we assume that the consideration of these other properties is abstracted from; we deal with them as having only these two properties, magnitude and direction.

3. Thus the motions of the different soldiers of a regiment marching straight forward through a field take place in different parts of the space of the field. These different motions are distinguished from each other by having different *positions* in the field. But abstracting from their property of *position*, they are vectors all having the same direction and the same magnitude, and are, therefore, all *one and the same vector.* Two coaches running along different but parallel roads and in the same direction, and with the same speeds along these roads, have velocity vectors which are equal or rather are the same vector, although the actual motions occur in different parts of the country.

Illustrations

Again, simple changes of position, velocities, and momenta are all vectors, but of different kinds, being distinguished one from the other by the first having *position* only as a property besides direction and magnitude, while besides these the second involves both *position* and *time*, and the third *position, time*, and *mass*.

Again, if two masses have the same momenta both quantitatively and as regards direction, these two momenta are one and the same vector although they have different positions and occur in different masses.

It is to be noted that vectors of different *kinds* cannot possibly be *added* together.

4. The sign $=$ indicates quantitative or numerical equivalence. The sign $\parallel$ ordinarily indicates simple parallelism, but we will, when applying it to vectors, use it invariably to indicate identity of direction. These two signs are combined into ⋕ to symbolise *vector equality* or combined quantitative equality and directional identity. This sign is to be used in all vector *equations*. The symbol ⧧ is used to indicate equality in magnitude and in parallel but opposite directions. The symbol ǂ indicates quantitative equality combined with parallelism without reference to direction or 'sense.'

Symbols of Equivalence

When it is desired to express equality, identity, or coincidence of the other properties of the vectors compared besides those of direction and magnitude, the sign ⫵ will be used. This sign will be more particularly used to express coincidence of *position* or situation as well as of direction and magnitude of the vectors compared. When we consider the results or effects of the combination of the three properties, position, direction and magnitude, the things dealt with will be called *locors*, and we may term ⫵ the symbol of locor equality. Nearly all vectors are actually locors; a vector is, usually if not always, simply a locor considered without regard to its definite position.

The sign ⫵̳ will indicate exact locor oppositeness; thus a ⫵̳ b means that a and b are locors with equal magnitudes and exactly opposite directions lying along coincident lines.

Locors

5. The locors or vectors that are chiefly dealt with in mechanics are the following :—

Difference of Position.

Change of Position or Displacement.

Velocity and Change or Difference of Velocity.

Acceleration (i.e. Time Rate of Change) of Velocity.

Momentum and Acceleration of Momentum.

Force and the one-directional aspect of a Stress, including the stresses called Gravitation, Cohesion, Electric and Magnetic Induction.

Flow of Fluid or Flow of any quantity of any kind whatever, and this being either the Integral Flow throughout a given time or the time-rate of flow, and either the Integral Flow through a given area or the flow per unit of area. The flow may either be of one or other kind of *substance* or matter, or it may be of a *condition*—for example, of a dynamic condition such as momentum, stress, strain, kinetic energy. Electric current may be taken as coming under the latter head. **Locors**

6. Rotations and Angular Velocities may also be represented by vectors or locors, the direction and position being taken along the axis of rotation and the quantity (represented to a convenient scale by a length measured along the axis) being an angle or angle per second. The length along the axis is to be directed + or – according to the direction of rotation; the most convenient convention being that it is to have that direction in which the rotation will appear *right-handed.* Viewed in the opposite direction, the rotation would, of course, appear left-handed. But there is a distinction between this kind of quantity and those previously mentioned which should not be lost sight of. While all the others are vectors in themselves and in reality, rotations are evidently not vectors at all; it is only the graphic representation of the rotation by a directed line that is a vector. The vector is the conventional symbol, not the real thing itself. This essential distinction should be preserved in the language used, and this kind of quantity will be called a '*rotor*,' a name applied by Professor Clifford to all locors as well as to what is here called a 'rotor.' When the position of the axis of the rotor is concerned in the problem dealt with, as well as the direction of that axis and the magnitude of the rotation, the rotor may be called a 'locorotor.' **Rotors**

Angular momenta and force-couples are rotors of a kind very similar to rotations and angular velocities.

7. In some circumstances it is convenient to represent plane surfaces similarly by directed lines. These may also **Surface rotors**

be called rotors. The line is taken of a length measuring to the scale adopted the area of the surface. It is taken normal to the surface, and directed in such a way as to indicate whether the area is to be counted positive or negative. An area is frequently to be regarded as generated by the motion of a line—i.e. as being swept out by a line, and the sign of the area then indicates the general direction in which it has been generated. Thus it may be a 'polar area' swept out by the radius vector from a fixed pole to a moving body. In this case the sign given to the area is naturally made to coincide with that of the rotation of the radius vector round the pole—i.e. the line representing the area is to be drawn in that direction in which the sweeping out of the area round the pole appears to be right-handed. There are several other conventions with regard to the signs of areas used under various circumstances. This rotor representation of a surface indicates nothing as regards the *shape* of the boundary of the surface, and, therefore, is not a complete description of the surface any more than the vector representation of a momentum is a complete description of the momentum; this latter not affording any information regarding the mass or the velocity separately of the moving body, but only regarding the product of the mass and of its velocity.

Surface rotors

8. We habitually talk of all these things having position and direction in *space*. It should be held clearly in view that in doing so we think only of *relative* positions; that the space referred to is that occupied by and geometrically attached to, so to speak, and surrounding, some material object or set of objects. Careful consideration will convince anyone that we have, and can have, no other idea of space than this relative one; that talk of *absolute* space and *absolute* position is talk only and does not correspond to any real consciousness or any real thought, probably because it quite certainly corresponds to no real experience. Clerk Maxwell says with keen and delicate sarcasm, 'Anyone who will try to imagine the state

Relativity

of a mind conscious of knowing the absolute position of a point will ever after be content with our relative knowledge.'[1]

9. The positions of things in space can only be recognised and defined in relation to other things. Thus the position of a picture is thought of in reference to the walls and furniture of the room in which it hangs; the position of a cricket-ball at any instant is thought of in relation to the wickets, bat, fielders, and the field on which the game is being played; the motion of an engine-crank is regarded as taking place in the space defined by reference to the engine bed-plate; the motion of a pinion in a watch is thought of in relation to the space defined by reference to, that is 'geometrically attached to,' the watch-case; the motion of a ship propeller-blade may be viewed as taking place in the space attached to the ship itself, or in the space attached to the water, or in the space attached to the solid earth, this latter being different from the water-space if there be a water current giving motion as between the water and the earth. The space geometrically attached to the body in relation to which a vector, locor, or rotor is defined is conveniently called the 'field' of that body. Thus, the stellar field is that in which the motion of the sun is astronomically calculated; the field of the sun is that in which the planetary orbits are mapped

The 'Field'

[1] *Matter and Motion*, p. 20. 'Absolute space is conceived as remaining always similar to itself and immovable. The arrangement of the parts of space can no more be altered than the order of the portions of time. To conceive them to move from their places is to conceive a place to move away from itself.

'But as there is nothing to distinguish one portion of time from another except the different events which occur in them, so there is nothing to distinguish one part of space from another except its relation to the place of material bodies. We cannot describe the time of an event except by reference to some other event, or the place of a body except by reference to some other body. All our knowledge, both of time and space, is essentially relative. When a man has acquired the habit of putting words together without troubling himself to form the thoughts which ought to correspond to them, it is easy for him to frame an antithesis between this relative knowledge and a so-called absolute knowledge, and to point our ignorance of the absolute position of a point as an instance of the limitation of our faculties. Anyone, however, who will try to imagine the state of a mind conscious of knowing the absolute position of a point will ever after be content with our relative knowledge.'

out; the field of the earth is that in which wind and ocean currents and railway trains, &c., are regarded as moving; the field of the locomotive frame is that in which most commonly the momenta, forces, &c., of the various parts of the machine are considered; the water flowing through a turbine may be considered to move either through the field of the blades, which field is rotating along with the blades in the field of the fixed casing, or it may be regarded as flowing through this field of the fixed casing—i.e. the field of the earth —the word 'fixed' here meaning motionless in the field of the earth. We hardly ever think of the motion of a watch-pinion as relative to any other field but that of the watch-case, because it is that motion alone that has any great importance as regards the construction and action of the watch. It was probably only after the discovery that the rate of a watch depends slightly on the position as regards the force of gravity and as regards the force of terrestrial magnetism, that any one ever thought of the motions of its wheels relatively to the earth field. On the other hand, the motions of a ship propeller reckoned relatively to the ship and relatively to the earth are equally important, and, therefore, the recognition of the relativity of the motion in this case has always been familiar to engineers. It is the same in the case of turbines.

The 'Field'

10. Since all vectors are thus relative to one or other field, there will be no occasion to use the phrase 'relative to.' If the vector is said to be 'through' a specified field, that will mean that it is to be regarded relatively to that field. If it is said to be 'past,' 'over,' or 'round' a specified material body, such as the bed-plate or frame of a machine, that will mean that it is reckoned relatively to the 'field of that body.'

'Through,' 'past,' 'over,' 'round'

11. It is now evident that every possible motion or other vector has a dual aspect; or rather that the phenemenon of which the given vector is one aspect is also capable of being viewed in an exactly opposite aspect. It is necessarily a dual phenomenon, capable of interpretation by one or other of two

Reciprocal duality

equal and opposite vectors. Thus, while the propeller-blade has a certain motion through the field of the water, the water has a simultaneous exactly equal but opposite motion through the field of the propeller-blade. While the pump-bucket is moving upwards in the field of the pump cylinder, the pump cylinder may be considered to be, and *actually is*, moving downwards in the field of the bucket. While the water of a river is flowing downwards through the field of the river bank, the bank is travelling upwards through the field of the water. While a crank is rotating right-handedly through the field of the engine bed-plate, the bed-plate is rotating with an equal left-handed velocity through the field of the crank. The ancient belief that the sun moved round the earth daily from west to east has exactly the same truth in it as the more common modern notion that the earth rotates daily about its own axis; these are only the two opposite aspects of one and the same phenomenon; the sun moves round the earth in the field of the earth, while the earth revolves in the opposite direction in the field of the sun, the two axes of rotation being coincident. The two experiments that will produce the greatest personal conviction of the truth of this doctrine of the *essentially* dual and relative nature of motion are—first, riding in a train side by side with another train moving through the field of the earth with only a slightly different speed from our own, and, secondly, going down and up a deep mine shaft with the cage moving at a considerable velocity. The unfamiliar sensations that are experienced when these experiments are first made are *not illusions*, as they are often thought to be; they are, on the contrary, the awakening of the mind to a consciousness of the fact that it has hitherto been resting in the habitual delusion that all motions necessarily take place in the field of the solid earth. These two experiments should be repeated until the sense of strangeness wears off and the mind becomes habituated to, grasps familiarly, and makes part of its permanent instinct the true notion of the dual relativity of all motion.

Reciprocal duality

All other vectors are similarly the one-sided aspects of dual phenomena, the complete vector description of each of which consists of a pair of equal and opposite vectors through two different fields. The student is left to trace out for himself this principle of dual relativity or reciprocal duality through the series of phenomena corresponding to the different kinds of vectors mentioned above.

Reciprocal duality

12. The graphic process of summation of vectors is almost self-evident. Since the vector properties do not depend on position, we may place the vectors where we please in order to effect their addition, provided only we keep their directions and magnitudes unchanged. They may, therefore, be placed so as to form a continuous chain, the beginning of each vector being placed so as to coincide with the end of the one preceding it in the chain.

Vector addition

Thus in Fig. 44 the five vectors A B, B C, C D, D E, and E F, or α, β, γ, δ, and ε, are arranged in this successive or tandem fashion.

Here the vectors are added one on to the other, and the total result of this addition is the same as, or equivalent to, a vector from A to F—i.e. with direction A F and magnitude measured by the length A F to same scale as that employed in plotting the separate vectors. This being the total result, the vector A F may be called the Vector Sum, or in symbolic language calling A F by the letter σ,

$$AF \# \sigma \# \alpha + \beta + \gamma + \delta + \varepsilon.$$

13. The addition may be effected in any order—that is, the vectors may be joined to each other successively in any order; in whatever order they be arranged or plotted, the sum A F is always the same. In the figure one other order is shown—namely, $\alpha + \delta + \gamma + \varepsilon + \beta$.

Distributive law

By the above vector equation it is not meant that the single vector σ is in all respects equivalent to the combination $\alpha + \beta + \gamma + \delta + \varepsilon$; nor that this combination is in all

respects equivalent to the combination $\alpha + \delta + \gamma + \varepsilon + \beta$. All that is meant is that, so far as *total result* or *sum* is concerned, these three are equivalent to each other, or rather that they are the same—that is, have the same vector sum. **Distributive law**

14. Although the correctness of this process of vector addition is, as already said, self-evident to one who has geometrical experience, it is by no means self-evident that the process corresponds to any actually occurring physical combinations of vectors. It remains to trace this real correspondence in the case of each of the kinds of vectors already mentioned. Because the correspondence does actually exist for one kind of vector, it by no means follows that it does so for others.

Let us see what it means in the case of displacements or changes of position. Firstly, suppose the displacements to be those of one and the same body occurring successively in point of time. I wish to go from my house to, let us say, a house in Harborne Road. I go along St. Augustine's Road a certain distance, represented in direction and length by α; I then walk along Hagley Road a distance given by β in length and direction; I then perform the motion γ along Norfolk Road, then δ along Augustus Road, and finally I go along Harborne Road in the direction ε, a distance measured to scale by the length of ε. The total result, so far as change of position is concerned, is my displacement from a house in St. Augustine's Road to one in Harborne Road. Or in a survey, the problem being to determine the distance and direction of one point F on an estate from another point A, the surveyor has to range out successive lines, α β γ, &c., of various directions and lengths. The result of his adding one line to another is to obtain a single line of definite length and direction from A to F. Here, although the surveying of the different lines has taken place successively in point of time, the lines or vector differences of position exist simultaneously and continually. **Physical meanings**

Suppose now the vectors to be the simultaneous change of position of the different soldiers of a regiment on the march.

They are marching straight northwards, say. The object of the officer in command is to bring soldiers northward. Let us consider the result of the whole company marching thirty yards. The movement of each soldier this distance is so much partial accomplishment of the officer's object. This thirty yards northwards multiplied by the number of soldiers moved is the measure of the total amount of his object accomplished in this interval. This total result can be represented graphically by plotting to a suitable scale all these distances of thirty yards successively along a straight line due north. The sum of the different movements occurring simultaneously in different parts of the field is thus represented by a straight line whose length is the sum of their lengths. The movement of the regiment taken as a whole could now be obtained by dividing the length of this line by the number of the parts of which it is the sum. The quotient will, of course, be the motion of one soldier, because in this particular case, where all the parts are moving together, the motion of the whole is the same as that of each part.

Physical meanings

15. Suppose now the company to be a skirmishing party, the different units of which move in different directions and different distances. At a given instant let the officer in command be situated at the point P in the field, and the soldiers at the positions A B C, &c. Let the vector differences of position PA, PB, PC, &c., be called $a b c$, &c. These vectors are partly northward and partly eastward. Let the unit A stand a_n north of P and a_e east of P; that is, let $a \mathrel{\#} a_n + a_e$. Similarly, let $b \mathrel{\#} b_n + b_e$ and $c \mathrel{\#} c_n + c_e$, &c. The *average* distance north from the officer of all the units of the company is, of course,

Mean position

$$\frac{a_n + b_n + c_n + \&c.}{N},$$

where N is the total number of units. Also, the average distance eastward from P is

$$\frac{a^e + b_e + c_e + \&c.}{N}.$$

If a vector be drawn from P, whose northward and eastward components are

$$\frac{a_n + b_n + c_n + \&c.}{N}$$

and

$$\frac{a_e + b_e + c_e + \&c.}{N},$$

the extremity of this vector will give the *average difference of position* of the whole company from that of the officer. This may be called the *centre* of the company and indicated by I, the vector P I being called i. Now find the vector sum of the vectors $a\ b\ c$, &c., by arranging them successively as in the last figure. The northward component of this vector sum is evidently $a_n + b_n + c_n +$ &c., and the eastward component $a_e + b_e + c_e +$ &c. These are each N times the components of the vector i. Therefore, i has the same direction as the vector sum $a + b + c +$ &c., and its length is $\frac{1}{N}$th of that sum. The average or central position of the company from P, as defined by the vector i, is, therefore, $\frac{1}{N}$th the vector sum of the position vectors from P to all the units of the company. This explains the meaning of the summation of position vectors.

Mean position

16. Now let these units smultaneously move distances represented in direction and magnitude by α, β, γ, &c., the officer remaining at P. The position vector from P to the new centre of the company is $\frac{1}{N}$th of the vector sum

$$(a + \alpha) + (b + \alpha) + (c + \gamma) + \&c.;$$

that is, it equals

Displacement of centre

$$\frac{a + \alpha + b + \beta + c + \gamma + \&c.}{N} \# \frac{a + b + c + \&c.}{N} +$$

$$\frac{\alpha + \beta + \gamma + \&c.}{N} \# i + \frac{\alpha + \beta + \gamma + \&c.}{N}.$$

Call the simultaneous motion of the centre η.

Since the position vector to the original 'centre' or aver-

age position was i, the motion of the 'centre' resulting from the motions of the units is

$$\eta \mathrel{+\!\!\!\!\!\!+} \frac{\alpha + \beta + \gamma + \&c.}{N};$$

that is, $\frac{1}{N}$th part of the vector sum of the motions of the units. This is the explanation of the meaning to attach to the vector summation of motion vectors occurring simultaneously in different portions of matter in the same field.

So far we have spoken of the positions and motions of persons or material bodies as if each had one definite position, and as if the various parts of each all performed one definite motion. The different parts of an extended body must evidently have different positions in whatever field they be reckoned. The motions of these different parts considered as locors *must* be different, and considered as vectors simply they *may* be, and in general *are*, different. When, therefore, we speak of the position of a body as a whole (that is, not considered as regards its parts), it must be understood that we refer to its *central or average position* as defined above; and when we speak of the displacement or motion of any kind of a body taken as a whole (that is, eliminating consideration of the differences of the displacements of its different parts) we mean the displacement of its central position or, more briefly, of its 'centre.' It is important to note that the smaller the body be the less must the difference be between the positions and motions of its centre and those of its parts, *provided the small body be not during the motion considered broken up and disgregated into parts far asunder, in which case the assemblage of parts constituting the body ceases to be a small body in at any rate the original sense.*

Displacement of centre

We have explained the finding the centre of a body made up of separate units. The finding of that of a continuous body is precisely similar. It is to be geometrically divided into a number of equal parts of such degree of smallness that

the differences of position and motion of the minute parts of these small parts may be neglected. These equal small parts are then dealt with as the separate units of the regiment in our first explanation. The 'equal' parts may be taken of *equal volume*; in this case the centre found will be the 'centre of volume' or 'volumetric centre.' A 'centre of area' will be found if the thing dealt with be a surface. The equal parts may also be taken of *equal mass*, that is, of portions having equal inertia; and then the centre obtained is the 'centre of mass' or 'centre of inertia.' As it is usually given the latter name, it will in general be indicated on the diagrams in this book by the letter I, the first in the word 'inertia.'

Displacement of centre

17. The displacement of the centre of mass is the displacement of the mass as a whole. This displacement multiplied by the whole mass is called the 'integral mass displacement.' This may be otherwise expressed thus. If the equal mass-parts into which the whole is divided be each $\frac{1}{N}$th part of the unit of mass, then the $\frac{1}{N}$th part of the vector-sum of all the displacement vectors of these parts is the integral mass displacement. Thus we see what meaning to attach to the addition of simultaneous mass-position and mass-displacement vectors in the same field.

Mass displacement

18. Suppose now P, the position from which the position vectors are measured, to coincide with I. Then the vector i or P I becomes zero. But the vector-sum of all the position vectors from P is N i where N is a purely numerical factor. Therefore, the sum of all the position vectors from I is zero. This furnishes a definition of the 'centre' I slightly different from that already given; namely, 'the centre I of a body or aggregation of bodies is that point from which the sum of the position vectors to all its equal parts is zero.' This will apply to both the volume-centre and the mass-centre.

Displacement from centre

Similarly, it may be seen that the vector-sum of the differences of displacement of all the equal parts from the dis-

placement of the centre is zero. Let the displacement of the centre be called η as before. We saw that

$$\eta \nparallel \frac{\alpha + \beta + \gamma + \&c.}{N}.$$

Now α being the displacement vector of the part A, the difference between it and η is $(\alpha - \eta)$. Thus the vector-sum of all the similar differences is

Displacement from centre

$$(\alpha - \eta) + (\beta - \eta) + (\gamma - \eta) + \&c. \nparallel \alpha + \beta + \gamma + \&c. - N\eta$$
$$\nparallel N\eta - N\eta \nparallel 0,$$

because

$$\alpha + \beta + \gamma + \&c. \nparallel N\eta$$

from above. This important truth is sometimes expressed by saying that 'the vector-sum of the displacements from (or measured relatively to) the centre is zero.'

It should be observed that all these propositions are equally true of solid, fluid, gaseous bodies, and disgregated assemblages of bodies like our planetary system.

19. Let us now consider possible displacements not simply as regards the 'centre' of the displaced body, but also as regards its various parts. It will be unnecessary at present to consider any bodies except such as are not changed in shape or in size during the displacement. So far as regards the motion of such a body, it will be the same as if the body were absolutely rigid—i.e. incapable of such change of shape and size. For shortness' sake, therefore, the bodies to be taken may be called 'rigid' bodies, which must be understood to be shorthand for 'bodies which are invariable in shape and size during the occurrence of the vectors considered.'

Displacement of rigid body

Suppose the body A B C D in Fig. 45 be displaced from the position 1 to position 2, the displacement being 'coplanar'—i.e. the displacement of all its points being parallel to one and the same plane. The displacement of A is $A_1 A_2$, and that of B is $B_1 B_2$, &c. Draw $B_1 B^1 \nparallel A_1 A_2$. If all parts of the body moved in same direction and through same distance

as A, then B would come to B^1. The body could then be brought into the actual position 2 by a rotation θ round an axis through A_2 and perpendicular to the plane to which all the actual displacements are parallel, the rotation bringing B from B^1 to B_2. The whole displacement may be looked upon as a combination or resultant of these two motions, the first $\#\, A_1 A_2$, in which all parts move uniformly in same direction and through equal distances, i.e. in which all suffer the same vector displacement, being called a motion of 'translation.' Similarly, the actual displacement could be looked upon as the resultant of a 'translatory' displacement of the whole $\#\, B_1 B_2$, which would bring A to A^1, and of a rotation θ round an axis through B_2; or of a translatory displacement $\#\, D_1 D_2$, which would bring A to A^{11}, combined with a rotation θ round the line through D_2 perpendicular to the above plane as axis. In these different ways of analysing the whole displacement, the translatory displacements are different and to each corresponds a different position for the axis of the rotation θ; but note that the rotation θ is through the same angle and in the same direction in each case. Also it is, of course, indifferent whether the rotation be thought of as occurring *after* or *before* the translation in point of time. If it be taken as occurring before it, then its axis must be through the *initial* position of the point whose translation is taken. Or the rotation and translation may be supposed simultaneous, the axis at each instant coinciding with the simultaneous position of the same point of the moving body. In fact, it is evident that the axis of the rotation has a definite position in the moving body itself or in its own field that moves along with it, and not in the field through which the displacement occurs. The displacement of the whole rigid body may thus be described to be the resultant of a translatory displacement $\#$ to the actual displacement of *any* point of it or of its own field, and of a rotation θ round an axis through *that same point*

Displacement of rigid body

of the body or its field, the axis being normal to the plane of the total displacement.

Rotational displacement

20. The translatory displacement taken may be of *any* point either in the body itself or outside it in its *own* '*field*.' There is always one point either in the body or outside it in its field whose displacement is zero, unless the whole motion be one of pure translation unaccompanied by rotation. This point can be found by bisecting the displacements of any two points such as A and B (bisect $A_1 A_2$ and $B_1 B_2$), and drawing from the bisecting points perpendiculars to the displacements (perpendicular to $A_1 A_2$ and $B_1 B_2$). The intersection of these perpendiculars is the point of no displacement. If the two displacements $A_1 A_2$ and $B_1 B_2$ be in the same direction, the perpendiculars are parallel and their intersection is non-existent. The axis, normal to the displacement plane through this intersection, when it exists, is called the 'axis of rotational displacement.' The whole displacement may be looked on simply as a rotation θ round this axis. This is the simplest mode of viewing the motion, but the above more general analysis is for practical purposes more commonly useful.

21. In the general case of non-planar displacement, i.e. the displacements of different parts *not* being all parallel to one plane, the corresponding analysis of the displacement of the whole body is that it may be looked on as the resultant of a translatory displacement $\#$ that of *any* one point in it, and of a rotation θ round an axis in a certain direction through that point.

Non-planar displacement

Whichever the point be whose displacement is taken as the translatory component, the rotary component θ will be of the *same magnitude* (i.e. of same angle) and its axis will have the same direction. The position of the axis of θ, of course, varies with the point chosen. In this general case of non-planar displacement, this axis is not perpendicular to the displacement vectors of the various points. These latter vectors are not parallel to each other nor parallel to one plane, and since

the axis retains the same direction in the field whatever point is chosen as above, it can evidently have no constant directional relation to these vectors. Any point A being so chosen, the direction of this axis can be found as follows from the displaced positions of A and of any two other points, say B and C. Let $A_1 B_1 C_1$ and $A_2 B_2 C_2$ be the original and displaced positions of these three points. Through B_1 and C_1 draw $B_1 B^1 ⋕ A_1 A_2$ and $C_1 C^1 ⋕ A_1 A_2$. Then $B^1 B_2$ and $C^1 C_2$ are the *relative* displacements of B and of C round or past A. Bisect each of these relative displacements $B^1 B_2$ and $C^1 C_2$, and through the points of bisection draw planes perpendicular to these same lines $B^1 B_2$ and $C^1 C_2$. Both these planes must necessarily pass through A_2 because of the rigidity of the body whereby we have $A_2 B_2 = A_1 B_1$ and $A_2 B^1 ⋕ A_1 B_1$, therefore $A_2 B_2 = A_2 B^1$, and similarly $A_2 C_2 = A_2 C^1$. The intersection of these planes is the axis sought for. It passes through A_2. We shall not find frequent use for this analysis in graphics because its graphic use involves a construction in three dimensions, and these are so much more tedious than constructions in two dimensions that in adopting them one finds that many of the characteristic advantages of the graphic method have been lost.

Non-planar displacement

22. There is in this problem always one set of points lying along a straight line parallel to the axis of rotation, either in the body itself or outside it in its field, whose displacements are all equal and in the direction of the axis of rotation, that is, along the line itself on which these points lie. If this set of displacements be taken in the above analysis as combined with the rotation θ, the motion is seen to be the same as that of a *screw*. The axis of the screw coincides with the line on which the above-mentioned points lie. The pitch of the screw is $\frac{2\pi}{\theta}$ times the displacement of these points—that is, the pitch is such that the axial screw motion corresponding to the angular motion θ equals this displacement. This is the

Screw displacement

simplest analysis of the general non-planar displacement of a rigid body. This line coinciding with the axis of the screw may be called the 'axis of screw displacement.'

Screw displacement

It must not be supposed that in this combination of translatory and rotary motions the vectors representing the translation and the rotation θ can be added together in the manner already explained for vector summation. They are vectors of different kinds: one represents a set of parallel locors, and the other is a rotor, and they are incapable of being added.

23. We have hitherto considered the addition of displacements occurring in one field only. Let us now consider the meaning of the addition of displacements occurring in one body in two different fields.

Simultaneous co-planar displacements

Suppose a body, say an engine-piston, suffers a translatory displacement in the field of a ship. Call this displacement in this field *a*. Suppose that the ship is also given a translatory displacement *b* in the field of the earth. The diagram, Fig. 46, may make the addition clearer. If the piston had no displacement in the field of the ship, it would be simply carried along with the ship, and would thus be given in the field of the earth the displacement *b*. It shares this displacement with the ship in the latter field, and besides this it suffers the former displacement *a* of its situation as regards the decks, bulkheads, &c., of the ship. Its total displacement through the field of the earth is thus the vector sum of the two vectors *a* and *b*. This is the evident result whether the two displacements take place successively in point of time or simultaneously. There is here no question of displacements due to rotation, because both motions are supposed translatory.

Take next the connecting-rod, which has a rotation accompanying its translation; and suppose also that the ship rotates in the field of the earth through an angle ε at the same time as it moves with translatory motion.

Referring to Fig. 47, let A indicate the cross-head pin

centre and B the crank-pin centre. The field of the ship, which is the same as that of the engine-frame, because they are rigidly bolted together, is sufficiently indicated in the diagram by drawing in the guide-bars and the engine centre line. The paper itself will indicate the field of the earth. The displacements in both fields will be supposed wholly co-planar. The rotation ε of the ship may be supposed due to rolling if it be a screw-propeller ship, or to pitching if it be a paddle boat.

The displacement of the connecting-rod in the field of, or more shortly '*over*,' the ship or engine-frame consists of the displacement a of the point A and the rotation σ. This brings A and B into the positions A^1 B^1 in the ship-field.

The displacement of the ship and its field over the earth consists of the displacement a_e of the point A_1 (the original position of A in this field) and the rotation ε. This brings the original positions of A and B in the ship from A_1 B_1 in the earth-field to $A_1{}^1$ $B_1{}^1$ in the same field. If there had been no displacement of the connecting-rod in the ship-field, the rod would now occupy the position $A_1{}^1$ $B_1{}^1$. But the displacement a_s over the ship has shifted A from $A^1{}_1$ to A_2, and round this centre A_2 the line A B, which has already been rotated through ε by being carried along with the ship in its motion over the earth, is further rotated through the angle σ by the rotation relatively to the ship. This line is thus brought into the direction A_2 B_2, and the length A B being taken along this line from A_2 determines the final position of B, namely, B_2, in the earth-field. In the illustration both ε and σ are taken positive—i.e. right-handed.

Simultaneous co-planar displacements

The result of this analysis may be shortly described thus: the displacement of the rod over the earth equals the displacement in the earth-field of any point A of the rod combined with a rotation $(\varepsilon + \sigma)$ round an axis through the same point A of the rod; the displacement of A being the vector-sum of a_e, the displacement *over the earth* of that point of the

ship which was coincident with the original position of the point A of the rod, and of a vector equal to a_s rotated through the angle ε where a_s is the displacement over the ship of the same point A of the rod.

24. In the diagram this last vector, namely a_s turned through ε, is marked $(-1)^{\frac{\epsilon}{\pi}} a_s$. This symbol may be used as representing the result of turning a through the angle ε, the symbol $(-1)^{\frac{\epsilon}{\pi}}$ implicitly indicating the direction of the axis round which the turning is to take place. The symbol $(-1)^{\frac{\epsilon}{\pi}}$ represents algebraically the operation of turning performed by the rotor ε. If the angle ε were $180° = \pi$, then $(-1)^{\frac{\epsilon}{\pi}} = -1$, and the operation would be a simple reversal of the vector. If the vector a were turned through $2\,\varepsilon$, or through $m\,\varepsilon$, m being any number, the result would be written $(-1)^{\frac{2\epsilon}{\pi}} a$, or $(-1)^{\frac{m\epsilon}{\pi}} a$. The same result could be obtained by two, or m, *successive* turnings of a round the same axis each through the angle ε. This latter process would have its result symbolised by $(-1)^{\frac{\epsilon}{\pi}}(-1)^{\frac{\epsilon}{\pi}} a$, or by a with $(-1)^{\frac{\epsilon}{\pi}}$ written before it m times. But according to the ordinary rule of multiplication and addition of indices, these expressions are equivalent to $(-1)^{\frac{2\epsilon}{\pi}} a$ and $(-1)^{\frac{m\epsilon}{\pi}} a$. This ordinary rule of multiplication, therefore, can be followed in finding the result of combining a number of successive rotations expressed according to the above algebraic convention. The graphic representation of this rotational operation has already been mentioned and will be dealt with in detail subsequently.

Hamilton's nomenclature

With this nomenclature the above analysis of the whole motion of the connecting-rod may now be further shortened into the following: the displacement of the rod over the earth

equals the displacement $\left\{a_e + (-1)^{\frac{\varepsilon}{\pi}} a_s\right\}$ of any point A of the rod combined with a rotation $(\varepsilon + \sigma)$ round an axis through the same point A in the rod.

In order the more thoroughly to explain this construction, we will now show the correctness of the result of applying it to solve the converse problem to find the displacement of the rod relatively to the ship when its displacement over the earth and the displacement of the earth past the ship are given. All displacements being purely relative, the two problems ought to be equally easy of solution. The result of compounding the two given displacements must be, of course, to give a resultant displacement through the ship-field from A_1 B_1 to A^1 B^1 in the last figure.

The data of the present problem consist, first, in the displacement over the earth of the point A of the rod from A_1 to A_2 (call this vector $A_1 A_2 \# \delta$), and its rotation in this earth-field round the same point A through the angle $B_2{}^1 A_2 B_2$ (call this angle θ); and, second, in the displacement of the earth past the ship. In terms of the data of the last problem we have

Hamilton's nomenclature

$$\delta \# a_e + (-1)^{\frac{\varepsilon}{\pi}} a_s,$$

and $$\theta = \varepsilon + \sigma.$$

The displacement of the earth past the ship might be given as the movement of the point of the earth originally at A_2 to A^1 combined with a rotation $-\varepsilon$ round A^1 as axis. But the original position of A of the rod in the earth-field is not A_2 but A_1. Therefore, in order to follow out the exact converse of the procedure in the last problem we must have the earth displacement past the ship defined by the movement of its point originally at A_1. Call this movement γ. Now this point would move $-a_e$ through the ship-field, if there were no rotation of the earth in this field; but, seeing that there is the rotation $-\varepsilon$, the displacement is $-a_e$ turned through

the angle $-\varepsilon$ in the ship-field. Thus, according to our previous nomenclature, $\gamma \mathrel{\#} (-1)^{\frac{-\epsilon}{\pi}} (-a_e)$.

Then according to the analogy of the result in the previous problem, namely, a displacement $\left\{a_e + (-1)^{\frac{\epsilon}{\pi}} a_s\right\}$ along with a rotation $(\varepsilon + \sigma)$, we obtain from the present problem a displacement of the point A of the rod $\mathrel{\#} \left\{\gamma + (-1)^{\frac{-\epsilon}{\pi}} \delta\right\}$ combined with a rotation $(\theta - \varepsilon)$.

Writing these as above in terms of the data of the first problem, the displacement becomes

Hamilton's nomenclature

$$(-1)^{\frac{-\epsilon}{\pi}} (-a_e) + (-1)^{\frac{-\epsilon}{\pi}} \left\{a_e + (-1)^{\frac{\epsilon}{\pi}} a_s\right\}$$

$$\mathrel{\#} (-1)^{\frac{-\epsilon}{\pi}} \left\{-a_e + a_e\right\} + (-1)^{\frac{-e+e}{\pi}} a_s$$

$$\mathrel{\#} a_s$$

because $-a_e + a_e = o$ and $(-1)^{\frac{-e+e}{\pi}} = (-1)^{\circ} = 1$; and the rotation becomes

$$\varepsilon + \sigma - \varepsilon = \sigma.$$

Thus the whole displacement of the rod through the ship-field consists in the movement a_s of the point A of the rod and the rotation σ round the same point A of the rod as axis.

It will be a useful exercise for the student to draw out for the last problem the diagram corresponding to Fig. 47 for the former problem.

Evidently this solution is equally true, whether the displacement of the body in the one field and the displacement of that field in the other field be simultaneous or successive in point of time.

25. It has been explained here only for co-planar motions. An exactly similar result can be obtained for non-planar motions in different fields. This case, however, can only be understood after closer consideration of the laws of the addi-

tion of rotors. It need not be given in this book because the graphic construction is not a convenient one; the problem is best solved by other than graphic methods.

Simultaneous non-planar displacements

What has been said regarding the addition of vector translatory displacements of points or rigid bodies applies equally to the vectors we have called 'Integral Volume Displacement' and 'Integral Mass Displacement.'

26. In none of these cases is there involved any consideration of *time-rates.* The changes occurring concern only the differences between the initial and final conditions of the quantities changed; the consideration of the intermediate conditions of position, &c., is entirely eliminated from the problem. Let us now see what the addition of time-rate vectors means. The simplest of these is linear velocity. The average velocity during the interval occupied by any displacement is simply the quantitative comparison between, or the quantitative measure of the physical relation between, the displacement and the time occupied. By analogy with purely numerical ratios between quantities of the same kind, it may be called the physical ratio between the displacement and the time, but it is not a pure ratio in the ordinary and strict sense of the word. When we talk of the velocity of a body at a certain instant we mean the average velocity during the very small interval of time referred to as that instant.

Time-rates

When the simultaneous vector displacements of the different parts of a body or assemblage of bodies are added together as previously explained, no difference in the process will be effected if each displacement is divided by the interval of time in which all have occurred. This interval of time being common to all, all are affected by this division in the same ratio. The summation of the vector velocities now gives the velocity of the 'centre' of the system multiplied by the number of equal parts into which it has been divided, and this vector velocity of the centre may be obtained by dividing the vector-sum by this number of parts. If the parts be of

Simultaneous Velocities

equal mass, the velocity of the centre of mass multiplied by the sum of the masses will be the integral vector momentum of the system.

Integral momentum

No physical meaning can be given to the vector summation of the velocities of the individual parts of a system if these do not occur simultaneously, but at different periods of time.

Difference of velocities

27. The vector *differences* of the simultaneous velocities of the parts are the velocities of these parts relative to each other. Thus in Fig. 48, if α β and γ be the velocities of the parts A B and C, and if from any point p chosen as pole, these vectors, α β γ, be drawn to any convenient scale; then the line marked $(\alpha - \beta)$, directed as shown by the arrow, is the velocity of A relatively to B; the velocity of B relatively to A being the exact reverse of this. Similarly, $(\gamma - \beta)$ and $(\gamma - \alpha)$ with the directions shown by the arrow-heads are the velocities of C relatively to B and of C relatively to A; the velocities of B and A relatively to C being the exact opposites of these.

Velocities in different fields

28. When a body has a translatory velocity in one field, and that field has a simultaneous translatory velocity through a second field; as, for example, when a portion of water flows with a certain velocity along a pipe in a locomotive, and that pipe at the same time moves with the locomotive with a certain velocity over the earth; then the velocity of the body through the second field is the vector-sum of these two velocities. For if in Fig. 49 the two velocities be α and β, then, calculating the displacement over the earth from the position O in any time t, by means of oblique co-ordinates parallel to α and β respectively, and calling these co-ordinates a and b, we have $a = \alpha t$, and $b = \beta t$, and therefore $\frac{a}{b} = \frac{\alpha}{\beta} =$ a constant ratio for different t's, so long as the velocities α and β are maintained constant, that is, during the interval of time, long or short, during which we reckon α and β to be the velocities of water through pipe and of pipe over earth. This ratio being constant, the locus of the displaced position is a straight line

coinciding with the vector $(\alpha + \beta)$ drawn through O, that is, the water moves along this line in the field of the earth; and as the length of this displacement along this line bears, by similar triangles, the constant ratio

$$\frac{\text{magnitude of } (\alpha + \beta)}{\text{magnitude of } \alpha}$$

to the length of the simultaneous displacement along the pipe (in the field of the pipe), therefore the velocity over the earth equals both in magnitude and direction the vector-sum $(\alpha + \beta)$.

Similarly, differences of simultaneous velocities through the same or different fields may be obtained graphically. Thus, if we know (see Fig. 50) α to be the velocity over the earth of the water entering a turbine, and β to be the velocity over the earth of the part of the turbine which the water is entering, then $(\alpha - \beta)$ is the velocity of the water through the turbine, that is, relatively to the turbine. **Velocities in different fields**

There is no physical meaning to be attached to the addition or subtraction of successive velocities of one body in different fields.

29. But the successive velocities of one body in the same field may be compared by means of their vector differences. These velocity changes may be called *time* differences to distinguish them from the simultaneous differences previously explained. If these differences be divided by the time during which they occur, the result of this comparison will be the average time-rate of change of velocity during that interval of time. **Change of velocity**

Thus if (see Fig. 51) any point A of a body have at one time a position A_1 and a velocity α_1; if it move from here along the curved path $A_1 A^1 A_2$ to the position A_2 in the interval t, and have then the velocity α_2; then, drawing from any pole p and to any convenient scale the vectors α_1 and α_2, the line marked $(\alpha_2 - \alpha_1)$ in the figure, directed as indicated by the arrow-head, is the change of velocity that has occurred

in the interval t. If this vector $(\alpha_2 - \alpha_1)$ be divided by t we obtain the average time-rate at which the velocity has changed during this interval. This time-rate is called the average velocity acceleration.

30. If the interval of time be taken small the change in the direction and magnitude of the velocity will be small in a corresponding degree, because there occur in nature no absolutely sudden changes of velocity. If a large number of velocity vectors occurring at successive small intervals of time be drawn from p, a continuous curve can be drawn through

Hodograph

their extremities. This curve is called the 'hodograph' of the motion. During any instant the velocity acceleration is evidently the linear velocity along the hodograph of the extremity of the velocity vector radiating from p. Thus the acceleration is continually tangential to the hodograph. Velocities reckoned from averages taken over only a minutely small interval of time may be called '*instantaneous*' velocities.

31. Velocity accelerations are thus vectors, and can be added in the usual vector way. The addition of simultaneous accelerations in different fields being very important, we give here Fig. 52 in order to show clearly the truth of the statement that 'the acceleration of the resultant velocity is the resultant or vector-sum of the accelerations of the component velocities.' Let there be three velocities a b and c added together in one body, which in a given time change from

Sum of accelerations

a_1 b_1 and c_1 to a_2 b_2 and c_2. Call $a_2 - a_1 \# \alpha$ and $b_2 - b_1 \# \beta$ and $c_2 - c_1 \# \gamma$.

From the end of b_1 placed as in the figure draw α. This leads to a point to which the vector from the end of a_2 is evidently b_1. From this point draw β; this will lead to a point to which the vector from the end of a_2 is b_2, because $b_2 \# b_1 + \beta$. From the end of c_1 as drawn in full line in the figure draw $(\alpha + \beta)$. This leads to a point to which the vector from end of b_2 is c_1, because the end of b_2 is $(\alpha + \beta)$

away from the beginning of c_1 in full line. To the last $(\alpha + \beta)$ plotted we now add γ, which leads, therefore, to a point to which the vector from the end of b_2 is c_2, since $c_2 \# c_1 + \gamma$. This point, therefore, gives $a_2 + b_2 + c_2$ from the point p, and it differs from $a_1 + b_1 + c_1$ by $\alpha + \beta + \gamma$. This is the proof of the proposition stated.

The acceleration of velocity of a point may often be conveniently split into two components: one along the line of motion, termed 'tangential;' the other normal to it, and called 'centripetal' or 'radial.' Referring to section 30 for the meaning of the 'hodograph,' and calling the velocity along the hodograph diagrammatic (as being only that of an imaginary point in a constructed diagram) to distinguish it from that of the actual motion of a real point now spoken of, the tangential acceleration in the real motion is clearly the component of diagrammatic velocity in the hodograph directed away from the fixed centre of the hodograph; and the radial acceleration is the component of the same diagrammatic velocity perpendicular to the radius of the hodograph. Let the linear velocity in the real motion be v, this being also the radius in the hodograph; and let the radius of curvature of the path of motion be R. **Radial and Tangential Accelerations**

The angular velocity round the centre of curvature is then $\frac{v}{R}$. This is also the rate at which the direction of v changes, and is, therefore, the angular velocity of the hodograph radius. The linear velocity of the end of the hodograph radius resolved perpendicularly to the radius is, therefore, $v \times \frac{v}{R} = \frac{v^2}{R} = \omega^2 R$, if $\omega = \frac{v}{R}$ be the angular velocity.

This centripetal acceleration may be graphically calculated by any of the multiplication and division constructions shown in Fig. 8, Chapter III. Three special constructions for this purpose are shown in Fig. 66, Chapter IX.

32. The introduction of a *mass* factor into any of the vectors we have dealt with alters none of the additive pro-

perties we have considered. Thus momenta, accelerations of momenta, forces, stresses, rates of flow of water mass, sand mass, or any other kind of mass are all subject to the same graphic laws of vector addition and subtraction as have already been explained and illustrated. These laws are so simple that the student runs little risk of making mistake. In applying these processes, however, to any new kind of vector not previously dealt with, it is very necessary to inquire carefully whether vector addition or subtraction has any real physical meaning in this new connection, and, if there be such a meaning, what it is exactly. The student should in no case be content to employ graphic processes without clearly and accurately understanding the physical interpretation to be put upon them.

Mass factor

33. Rotors are in some cases added in the same way as vectors, but only in special cases. Displacement rotors cannot be added in this way; velocity rotors can. The difference arises from the fact that a velocity may be taken as a function of an extremely minute interval of time, or of an extremely minute displacement. Displacements are not in general minute. The vector mode of addition may be applied to displacement rotors provided they are extremely small. On the other hand, this process of addition cannot be applied to velocity rotors if these are *average* velocities through an interval of time or through a displacement *not* extremely small. In fact, the general graphic formula for rotor addition reduces to coincidence with that for vector addition when the rotors to be added are minutely small. Thus the velocity rotors which may be added as vectors may be called 'instantaneous' as distinguished from those that are obtained by taking averages over an interval of time longer than an instant.

Displacement rotors

Suppose that, in Fig. 53, A B represents a rod which suffers first a rotation α round an axis (supposed perpendicular to the paper) through A, this rotation displacing the

point B of the rod to B′, and which subsequently is rotated β round a parallel axis drawn through B in its new position B′.

Bisect the angles $\alpha = \mathrm{B\,A\,B'}$ by A b and $\beta = \mathrm{A\,B'\,A'}$ by B′ a, and let these bisecting lines intersect in P. Then $\mathrm{A\,P} = \mathrm{A'\,P}$ and $\mathrm{B\,P} = \mathrm{B'\,P}$, and, therefore, the actual double rotational displacement might be produced by a single displacement round a parallel axis through P. The angle $\mathrm{A'\,P\,A} = \mathrm{B'\,P\,B}$ because the first is double a P A and the second is double b P B′, while these two last equal each other, being formed by the crossing lines A b and a B′. But each of these last equals the sum $\mathrm{P\,A\,B'} + \mathrm{P\,B'\,A}$. Now $\mathrm{P\,A\,B'} = \frac{1}{2}\alpha$ and $\mathrm{P\,B'\,A} = \frac{1}{2}\beta$. Therefore $\mathrm{A\,P\,A'} = \mathrm{B\,P\,B'} = \alpha + \beta$. Thus a single rotation $(\alpha + \beta)$ round the axis through P would produce the same whole displacement as the two successive rotations, and this, therefore, may be called the resultant or the rotor sum of the two component rotors.

The position of P is most simply defined as the intersection to two lines through A and B making the angles $\frac{1}{2}\alpha$ and $-\frac{1}{2}\beta$ with A B and B A. Considering the exact reversal of the whole process bringing the rod back from A′ B′ to A B, the same axis P can be obtained by drawing through B′ and A′ two lines making the angles $-\frac{1}{2}\beta$ and $+\frac{1}{2}\alpha$ with B′ A′ and A′ B′. In this reverse process the rotation $-\beta$ must first be performed round B′ and then $-\alpha$ round A. **Displacement rotors**

If the rotation β had been performed round B *first*, this would have brought A to A″, and if the rotation α had then been performed round A″ the new position of A B would have been brought to B″. The rotor sum would then be a rotation $\beta + \alpha$ round P″ instead of round P; P″ being found by drawing from B and A two lines making the angles $+\frac{1}{2}\beta$ and $-\frac{1}{2}\alpha$ with B A and A B.

Thus in this case the axis of the rotor sum $\alpha + \beta$ (α being first in point of time and β subsequent to α) has not the same position as the axis of the rotor sum $\beta + \alpha$. Notice that here the axes are defined as lying in given positions in the moving

body and are carried along with the body through the field in which the rotation takes place.

In this case the magnitude $(\alpha + \beta) = (\beta + \alpha)$ of either of these rotor sums $(\alpha + \beta)$ or $(\beta + \alpha)$ equals the arithmetic sum of the magnitudes of the component rotors. This results from the parallelism of the two axes.

Now suppose that the two axes of rotation are given fixed in position in the field through which the rotations take place, not in the field of the moving body.

Displacement rotors

In Fig. 54, let A be the axis in the field of rotation round which the first rotation α takes place. This brings point B of the body to B′, B being the position in the field round which the second rotation β takes place. This second rotation round B brings point A of the body to A′ and the point originally at B but now at B′ from B′ to B″ The axis of resultant rotation is to be found by bisecting line A A′ and drawing through the bisecting point a perpendicular to A A′; drawing a perpendicular to B B″ through the middle of B B″, and taking the intersection P of these two perpendiculars. But this last perpendicular passes through A′, because the triangle B A′ B″ is simply the triangle B A B′ turned round through β, and in this latter triangle B A B′ the sides B A and B′ A are equal, so that the perpendicular to B B′ through the middle of B B′ passes through A. The perpendicular through A on B B′ makes the angle $\frac{1}{2}\alpha$ with A B. Therefore the line A′ P makes the angle $\frac{1}{2}\alpha$ with A′B. Also, since P B bisects angle A′ B A, and since A′ B = A B, therefere angle P A B = P A′ B = $\frac{1}{2}\alpha$. Thus B A P = $-\frac{1}{2}\alpha$. Therefore, the position of the axis P is found by drawing through A and B lines making angles $-\frac{1}{2}\alpha$ and $+\frac{1}{2}\beta$ with A B and B A.

Similarly, if β round B were performed first and α round A second, the axis P″ of the rotor sum $(\beta + \alpha)$ would be found by drawing through B and A lines making angles $-\frac{1}{2}\beta$ and $+\frac{1}{2}\alpha$ with B A and A B. The two axes P and P″ are placed symmetrically on opposite sides of the line A B at equal distances

from that line. The angle of the rotor sum in either case is $(\alpha + \beta)$.

It is evident that the axis of the rotor sum $(\alpha + \beta)$ with the axes of α and β fixed in the field of the rotating body coincides with that of the rotor sum $(\beta + \alpha)$ with the axes of α and β fixed in the field through which the rotation takes place coincidently with the initial positions of the previously mentioned axes in the body. Again, the axis of $(\beta + \alpha)$ with axes of β and α fixed in the body coincides with that of $(\alpha + \beta)$ with the axes of α and β fixed in the field of rotation.

In all these cases the distances of the resultant axis P from the component axes A and B have the ratio

$$\frac{\text{A P}}{\text{B P}} = \frac{\sin \frac{1}{2} \beta}{\sin \frac{1}{2} \alpha}.$$

If the angles α and β are of opposite signs and nearly equal in magnitude, the resultant axis is very far off. If α and β are exactly equal and of opposite signs, the displacement becomes one of translation alone. **Displacement rotors**

There is still a third case really more important than either of the others—namely, that in which one axis is fixed in the field through which the motion occurs, and the other axis fixed in the moving body. For instance, if a crank revolves on an axis fixed in the field of the earth, and carries with it on a crank-pin a wheel or a rod which revolves round this pin through a given angle relatively to the crank, then the motion of the wheel over the earth corresponds with the third case of the problem now being stated. Referring to Fig. 53, it will be seen that the position of the resultant axis in this case is precisely the same as in that figure, with the proviso that the axis fixed in the field must be considered as the first taken in the problem of that figure. There is, in fact, in this third case no question of the order of the rotations; the rotor sum of the component rotors is the same in whichever order they be taken. If A be the axis fixed in the field and α the rotation round it, and if B be the axis fixed in

the displaced body and β be the rotation round B, then the place of the resultant axis is found by drawing through A and B two lines making angles $\frac{1}{2}\alpha$ and $-\frac{1}{2}\beta$ with A B and B A. This position is the same whether the rotations be performed in one or the reverse order, or if they be simultaneous. The rotor sum is a rotor round this resultant axis equal to $(\alpha + \beta)$ in magnitude.

34. Suppose now the axes to be inclined to each other, and to intersect. We have here also the three cases distinguished as above with precisely similar variations in the result. In the first two cases the position of the resultant axis depends on the order in which the rotors are taken; in the third case it is independent of that order; the magnitude or angle of the rotor sum is the same in all three cases, and does not depend on the order of rotation.

In Fig. 55 let C A and C B be the two axes intersecting in C. Diagrams I. and II. are two views at right angles to each other, the first (I.) being taken backwards along the axis of α,

Displacement rotors

that is, from A towards C, or in the negative direction along this axis; the second (II.) being taken in the direction perpendicular to both C A and C B, that is, the plane of (II). coincides with that containing the two axes of α and β. Imagine a sphere of unit radius described about C, and suppose A and B be the points where the axes of α and β intersect the surface of this sphere. The great circle perpendicular to C A is marked in both views *a a a*. It appears a straight line in II. and a circle in I. The great circle perpendicular to C B is marked *b b b*, being a straight line in II. and an ellipse in I. α and β are taken both positive—i.e. right-handed, as viewed in the positive directions along their axes. In (I.), α and β appear left-handed because the view is taken in the negative direction as regards both axes.

Through C A draw two planes making angles $\frac{1}{2}\alpha$ on the opposite sides of plane A C B. Through C B draw two planes making angles $\frac{1}{2}\beta$ on opposite sides of the same plane B C A.

Let these intersect in C P and C P′. These lines C P and C P′ are the resultant axes occurring in the three cases of the problem. For a line in the body originally coincident with C P would be shifted to C P′ by the rotation α round C A, and would be shifted back again from the position C P′ to C P by a subsequent rotation β round C B. These two rotations taken in this order, therefore, leave finally the line C P without displacement, and C P is therefore the resultant axis of α followed by β, the axes being fixed in the field. C P′ is the resultant axis of α followed by β if the axes be fixed in the displaced body.

Similarly, the rotations β and α taken in this order round C B and C A would shift a line originally coincident with C P′ first to C P and then back again to C P′. This last is, therefore, the resultant axis if the component axes be fixed in the field.

Suppose the plane C P B to be fixed in, and moved along with, the body. The first rotation α round C A would bring this plane into the position C P′ B′ where the angle B′ P′ B equals double the angle B P a. After the second rotation which swings B P′ round C B back into the position B P, the above plane will make with B P C (that is, its original position) this same angle B P′ B′ or double a P B. This angle is, therefore, the rotor magnitude of the rotor sum $(\alpha + \beta)$, and its direction is right-handed or positive round the resultant axis C P. The angle a P B is the supplement of B P A, and double a P B is, therefore, one whole revolution minus double B P A. Since one whole revolution leaves the resultant displacement zero, a revolution less $2 \times$ B P A taken right-handedly gives the displacement the same as would be effected by a left-handed rotation $2 \times$ B P A. The total rotation may thus be considered $-\ 2 \times$ B P A, but for many purposes it seems better to write it $\{2\pi - 2 \times \text{B P A}\}$, which indicates more unambiguously the real magnitude and direction of the angle through which the body has been turned.

Displacement rotors

The arc or angular distances of the resultant axis from the component axes have the ratio

$$\frac{\sin \mathrm{A\,C\,P}}{\sin \mathrm{B\,C\,P}} = \frac{\sin \frac{\beta}{2}}{\sin \frac{\alpha}{2}}.$$

This result can be graphically represented in another way, which shows a more evident analogy with vector summation.

Draw C p' perpendicular to plane B C A. Draw C b_1 perpendicular to plane A C P. Thus angle b_1 C $p' = \frac{1}{2}\alpha$. Draw C a_2 perpendicular to plane P C B, thus making angle p' C $a_2 = \frac{1}{2}\beta$. Make the angles b_1 C $p_1 = b'$ C $p' = p'$ C $b_1 = \frac{1}{2}\alpha$ in the plane perpendicular to axis C A. Make angles a_2 C $p_2 = a'$ C $p' = p'$ C $a_2 = \frac{1}{2}\beta$ in plane perpendicular to axis C B. On the surface of the sphere make the angle $b_1\, p_1\, a_1$ = angle $b'\, p'\, a' = b_1\, p'\, a_2$, and the arc $p_1\, a_1 = p'\, a' = p_2\, a_2 = \frac{1}{2}\beta$. Draw a great circle arc from a_1 to b_1. Suppose the spherical triangle $a_1\, p_1\, b_1$ to be fixed in and moved along with the rotated body. The first rotation α round C A will bring this triangle into the position $a'\, p'\, b'$ because angle p_1 C $p' = \alpha$. The second rotation β round C B will bring the side $a'\, p'$ of this triangle to position $a_2\, p_2$ because angle p' C $p_2 = \beta$. Suppose the triangle now to stand in the position $a_2\, p_2\, b_2$. Draw a great circle arc from b_1 to a_2. Comparing the triangles $a_1\, p_1\, b_1$ and $a_2\, p'\, b_1$, since the arc sides $a_1\, p_1 = a_2\, p'$ and $p_1\, b_1 = p'\, b_1$ and the angle $a_1\, p_1\, b_1 = a'\, p'\, b' = a_2\, p'\, b_1$; therefore, the arc $b_1\, a_2 = b_1\, a_1$ and the angle $a_1\, b_1\, p_1 = a_2\, b_1\, p'$, and $b_1\, a_2\, p^1 = b_1\, a_1\, p_1 = b_2\, a_2\, p_2$. Therefore, the three arcs $a_1\, b_1$, $b_1\, a_2$, and $a_2\, b_2$ all lie on the same great circle. Thus the triangle could be brought from its first to its final position by a single rotation through the angle a_1 C $a_2 = 2 \times b_1$ C a_2 round an axis perpendicular to the plane b_1 C a_2. It can be shown that this last axis coincides with C P previously found.

Displacement rotors

Thus if the component rotations be represented to half

size (i.e. $\frac{1}{2}\alpha = \alpha$) by arcs on great circles perpendicular to their axes, and these arcs be placed joining each other so as to form two sides of a spherical triangle, taken in the order in which the rotations occur, the great circle arc forming the third side of this triangle, taken in the direction from starting point of the first rotation arc to end of second rotation arc, will represent the true direction and magnitude (this also to half size) of the resultant rotation or rotor sum, the resultant axis being perpendicular to this last arc. **Displacement rotors**

By this last construction any number of rotors round axes intersecting in one point can be added by a precisely similar process to that of vector summation, there being substituted for the straight line representations of the vectors great circle arcs on a sphere of unit radius to represent the rotors.

35. When the rotors have minutely small displacement-magnitudes, their arc representations in the last construction become practically straight lines. In this case the law of rotor summation becomes identical with that of the addition of vectors. The vector method of addition can be applied to the lines taken along the axes of the rotors to represent both magnitude and direction of the rotors; because, in the case of adding two rotors of minute magnitude, evidently the axis of their sum lies in the same plane as the component axes, the three arcs to which these axes are perpendicular all lying on a minutely small spherical surface, i.e. practically in one plane. **Small rotors**

36. Angular velocities are rotors referring to minutely small displacements, and are therefore subject to this graphic law. Angular momenta, accelerations of angular velocity or of angular momenta, and force couples are other examples of rotors all falling in this class and all to be dealt with graphically by this vector method of summation. **Angular velocities**

When the rotor magnitudes refer to minutely small dis-

placements the angular distances of the resultant axis C P from the component axes C A and C B have the relation

Angular velocities

$$\frac{\sin \mathrm{A\,C\,P}}{\sin \mathrm{B\,C\,P}} = \frac{\beta}{\alpha}.$$

If the axes be also parallel, the linear distances of these axes have the ratio—

$$\frac{\mathrm{A\,P}}{\mathrm{B\,P}} = \frac{\beta}{\alpha}.$$

CHAPTER VIII.

LOCOR ADDITION AND MOMENTS OF LOCORS AND OF ROTORS.

1. We have already seen how to find the average, mean or 'central' position of a number of points, mass-particles, or volume-particles. The process consists in adding the vectors from an origin to the different units and dividing the vector sum by the number of units.

We have also dealt with the problem to find the mean or central position of a number of parallel locors. The solution consists in multiplying each locor by its perpendicular distance from a datum axis, adding these products, which are called the moments round this axis, and dividing the sum of the moments by the sum of the vector magnitudes. The graphic execution of this process has been fully explained in Chapter VI., and from what follows it will be seen that it is only a special case of a general construction applicable to all locors, whether parallel or not.

Locor moment

The moment of a locor round an axis perpendicular to it is the physical product of the locor and of its perpendicular distance from that axis. The magnitude of the moment is the arithmetical product of the magnitudes of the two factors in the product. The unit of the product is a quantity having a definite rotational direction round an axis which has definite direction and position. A locor-moment is, therefore, a rotor. But in the moment-product all indication of the definite position or direction of the locor is lost. From the moment

one can only deduce that the locor has one of the infinite number of possible directions and positions perpendicular to the axis of the moment.

2. While locor moments thus afford no indication as to the positions of the component locors, still they afford means of finding the average or central position of a number of locors. If the moments round any axis chosen as a datum be summed up, and the moment sum be divided by the *vector sum* of the locors, a position is obtained defined by a distance from the axis and by parallelism to the vector sum. A locor taken at this distance from the axis, parallel and equal to the vector sum, will have a moment round the axis equal to the above sum of moments. This locor may be looked on as the sum of the given locors both with regard to moments and with regard to vector sum. This equivalence with regard to moment and to vector sum is expressed by the sign ⧻.

Central locor position

3. The locor sum of two locors whose lines intersect lies in the line passing through this intersection. Thus in Fig. 56 let A B and B C be the two locors, and P the projection of the axis. From C draw C D ⋕ A B. Then B D ⋕ A B + B C; and since B D considered as a locor lies through the intersection of the lines of A B and B C, the proposition is that B D ⧻ A B + B C. The magnitude of the moment of A B round P is double the area of triangle A B P; that of B C round P is double the area of triangle B C P; and that of B D round P is double the area of triangle B D P. The two triangles A B C and C D P having their bases A B and C D parallel, the sum of the heights of their vertices C and P above these bases equals the height of P above A B. Their bases being also equal to each other and to that of A B P, the area A B P = area A B C + area C D P.

Intersecting locors added

= area C D B + area C D P.

Add area B C P to each side of this equation and note that B C P + C D B + C D P = B D P. We find area A B P + area B C P = area B D P. Therefore, taking moments round P,

moment of AB + moment of BC = moment of BD, or

$$AB + BC \mathrel{\#\!\!\!|} BD.$$

The proof of the proposition is independent of the position of P; it is true for every possible axis. Thus BD is the locor sum independently of, and without reference to, any special axis.

Resolution

4. From this it becomes evident that any locor may be split into component locors at any point of its line, the magnitudes and directions of the components having the same relations to those of the resultant as in the case of vectors, and the component and resultant locors having their lines all intersecting in one point.

Axial component

5. The above refers only to the case of components and resultant lying all in one plane perpendicular to the axis. If a locor be parallel to an axis, it has no rotational direction round that axis, and it is, therefore, said to have no moment, or zero moment, round that axis. If now a locor be oblique to an axis, the locor may at any point of its length be split into two components, one parallel to the axis and the other perpendicular to it. The former component has zero moment, so that the whole moment of the locor equals that of its latter component perpendicular to the axis. This convention or system of calculating moments corresponds exactly with physical facts, as, for example, the total turning power or influence round an axis of a number of different forces. It also enables the above proposition regarding locor summation, or composition and resolution, to be stated perfectly generally, no matter into how many components the resultant may be split up, or whether they all lie in one or different planes. If S be the sum, and A B C D E be five components into which it is split up, such that the vector equation

$$S \mathrel{\#} A + B + C + D + E$$

is satisfied, and if A, B, C, D and E be taken as lying along lines all passing through *one* point in the line of S, then, looking on these as locors, the equation

$$S \mathrel{\#\!\!\!|} A + B + C + D + E$$

will be also satisfied, no matter which point on the line of S be chosen as the common intersection; the meaning of this last equation being that the moment of S round any axis whatever (perpendicular, parallel, or oblique to S) will equal the sum of the moments of A, B, C, D and E round the same axis.

Moments of forces, &c

6. Thus a force F may be resolved into three rectangular components $F_x F_y F_z$. If these three be taken as all acting along lines through any point in the line of F, the sum of their three moments round any axis whatever equals that of F round the same axis. Similarly with a velocity V, or an acceleration of velocity; with a momentum or an acceleration of momentum, or any other kind of locor.

7. If there be a number of locors not meeting in one point, the following graphic process gives their sum or resultant very easily. Using the notation explained for parallel locors at end of Chapter VI., let in Fig. 57 the spaces A, B, C, D, E be separated by the lines of the locors A B, B C, C D, D E, whose magnitudes and directions are given in the vector diagram *a b c d e*; the vector sum being *a e*.

Co-planar locors added

At any point in the line of A B resolve this locor into two component locors A P and P B chosen in any two directions. In the vector diagram draw through *a* and *b* the two lines $ap \parallel AP$ and $bp \parallel BP$. Then the magnitudes of these two components in the chosen directions A P and P B must be *ap* and *p b*. These two A P and P B may be taken as a locor substitute for the single locor A B. Now combine the locor P B with B C. The resultant is $PC \# pc$ and taken through the intersection of P B and B C. Since $AP + PB \# AB$ and $PB + BC \# PC$; therefore,

$$AB + BC \# AP + PB + BC \# AP + PC.$$

Next add locor P C to C D. The sum is $\# pd$; and if P D be drawn $\parallel pd$ through the intersection of P C and C D, this sum will lie along the line P D. Then we find

$$\mathrm{AB} + \mathrm{BC} + \mathrm{CD} \;\#\; \mathrm{AP} + \mathrm{PC} + \mathrm{CD} \;\#\; \mathrm{AP} + \mathrm{PD}.$$

Similarly, from the intersection of PD and DE draw the line $\mathrm{PE} \parallel pe$, the vector sum of pd and de. Take along this line a locor $\# \, pe$. If this locor be called PE, we have $\mathrm{PE} \;\#\; \mathrm{PD} + \mathrm{DE}$. Therefore,

$$\mathrm{AB} + \mathrm{BC} + \mathrm{CD} + \mathrm{DE} \;\#\; \mathrm{AP} + \mathrm{PD} + \mathrm{DE} \;\#\; \mathrm{AP} + \mathrm{PE}.$$

We have now got AP + PE as a locor substitute for AB + BC + CD + DE. But we can combine AP + PE into one locor $\mathrm{AE} \;\#\; ae$ and taken along the line AE drawn $\parallel ae$ through the intersection I of AP and PE. Thus the single locor AE, whose magnitude and direction are those of the vector sum ae, is the locor equivalent or sum of AB, BC, CD, DE. By this statement it is meant that the moment of AE round any axis whatever is equal to the sum of the moments round the same axis of AB, BC, CD and DE, the locor sum having also the same magnitude and direction as the vector sum of these same four.

Co-planar locors added

The space P is called the chain or the 'pen,' or 'single pen' (P) ABCDE, and the lines drawn from the pole p are called the 'pencil' (p) $a\,b\,c\,d\,e$. The whole process may be thus indicated: 'draw the single-pen (P) ABCDE parallel to the pencil (p) $a\,b\,c\,d\,e$.'

The construction as explained involves the assumption that all the locors lie in one plane. But if they do not do so, but are still all parallel to one plane, it will still apply so far as concerns moments about any axis perpendicular to that plane, because the moment of any locor round such an axis is not affected by its distance above or below the plane.

8. The construction is, however, not directly applicable to the summation of locors not all parallel to one plane. In this case the locors may be dealt with in either of two modes. In the one construction the locors are each to be resolved into two component locors. The resolution is to take place in each case at the point in which the locor line cuts any plane that

Non-planar and non-parallel locors

may be chosen as convenient for the purpose of this resolution, the same plane being adhered to for all the locors. The one component is to lie wholly in this plane; the other is to be perpendicular to it. This furnishes two sets of locors. The first of these sets lie all in one plane and can be summed up by the construction already shown. The second set are all parallel but are not co-planar. They are to be reduced to a single resultant as explained later in this chapter (v. section 15, Fig. 58 *b*). If the lines of these two resultants intersect, the two can be added so as to give a single locor as the complete locor sum of the whole given set. If they do not intersect, it is impossible to reduce the system to any single equivalent locor. Reduced to its simplest form, its equivalent is either the above-mentioned pair of non-parallel and non-intersecting locors, or else a single locor combined with a rotor. The method of reduction to this last form is given at the end of the present chapter.

Non-planar and non-parallel locors

The second mode of dealing with this problem in locor summation is to form three orthogonal projections of the given set of locors. Projections on any three different planes will serve, but, especially in engineering practice, three planes at right angles to each other are most convenient, and these may be called the 'front elevation,' the 'side elevation,' and the 'plan.' The plan projections of the locors form a set all parallel to one plane, and by the construction already given can be reduced to a single locor parallel to the same plane, which single locor will be their equivalent so far as vector sum and so far as total moment round any axis normal to that plane are concerned. This plan resultant is not the total resultant of the whole set of locors. The whole is equivalent to this plan resultant taken at a definite distance from the plane of the plan combined with a locor perpendicular to the plan taken in a definite position. This latter locor will be represented by a point in plan, and the position of the point depends on the

height above the plane of the plan at which the plan resultant is supposed to act.

The front elevations and side elevations of the locors are treated similarly. Each is by the ordinary process reduced to a single resultant. In the front elevation the whole system of locors is represented by the front elevation resultant and a point; and similarly in the side elevation. It remains to determine the proper positions of the points in each of the three views.

In what follows the three views are referred to by the letters (π), plan; (ε) front elevation; and (σ) side elevation.

In Fig. 58 *a* the three partial resultants are ρ_π ρ_ϵ and ρ_σ, supposed to have been obtained from the projection components of the several locors by means of suitably chosen poles and single pens drawn with sides parallel to the pencils radiating from the poles. The components of ρ_π and ρ_ϵ measured horizontally on the paper are equal; those of ρ_ϵ and ρ_σ measured vertically on the paper are equal; and that of ρ_π measured on the paper vertically equals that of ρ_σ measured horizontally. The ends of ρ_ϵ are placed directly above those of ρ_π and on the same level as those of ρ_σ merely for convenience; they are not necessarily so.

Non-planar and non-parallel locors

O E, O S and O S′ may be called ground lines; O V the plumb line. O S and O S′ represent the same line, and it may be supposed to be a *level south* line. O E may be thought of as a *level east* line; and O V as a vertical line.

The plan resultant ρ_π may be looked upon as having been obtained by resolving into a horizontal and a vertical component each of the group of locors at that point of its line of action where it cuts a given horizontal plane, say at the level *a b*. All the horizontal components then lie in one plane and their resultant ρ_π lies in the same plane, viz. at the level *a b*. The whole group of locors is now equivalent to ρ_π at the level *b a* and a single vertical locor, the position of which is to be determined. In elevation (ε) this system of two locors may

be considered as three by resolving ρ_π, which lies along $b\,a$, into two; one along $b\,a$ parallel to plane of (ε), one perpendicular to same plane (i.e. really $\parallel$ O S) through some point in $b\,a$, and a third vertical $\parallel$ O Y. Since the second is a point only in (ε) it contributes nothing towards the formation of ρ_ϵ. Therefore, since the vertical component together with that along $b\,a$ gives ρ_ϵ, the former must act through a where $b\,a$ meets ρ_ϵ; i.e. in (ε) it lies along line $a\,c$ and in (π) it is represented by some point in the same line $c\,a$. Now in (σ) the locor sum of the same set is represented by ρ_σ and a point. This point represents the component $\parallel$ O E, viz. that lying along $b\,a$ in (ε); and in (σ) this contributes nothing towards the formation of ρ_σ, which is the locor sum of the vertical component and the horizontal component $\parallel$ O S in (π), or $\parallel$ O S′ and along the line $a\,b$ in (σ). The vertical component must, therefore, act in (σ) through b where $a\,b$ meets ρ_σ. Measure the distance of b from O V in (σ) and plot it off in (π) from O E along the line $a\,c$. This gives the point d in plan through which the vertical locor lies. The magnitude of this latter is the vertical (or O V) component of ρ_ϵ or ρ_σ. We may call this ρ_v, naming the two other components of the resultant parallel to O E and O S by the letters ρ_e and ρ_s. The whole given group of locors is now reduced to the horizontal $\rho_\pi = \sqrt{\rho_e^2 + \rho_s^2}$ at the level of a in (ε) and the vertical ρ_v acting through d in plan.

Non-planar and non-parallel locors

The position of d depends on the level chosen for ρ_π, namely, that of a. It is easy to recognise that the locus of the different positions of d is the straight line $d\,d' \parallel \rho_\pi$.

If now it is desired to represent the whole group in (ε) by a single locor ρ_ϵ parallel to plane of (ε) and another ρ_s perpendicular to the same, the distance in front of the plane of (ε) at which ρ_ϵ is to lie must be chosen. In (σ) set this distance off $\parallel$ O S′ from O V and draw a line $\parallel$ O V at this distance cutting ρ_σ in e. Set the same distance off in (π) $\parallel$ O S from O E, and at this distance draw a line $\parallel$ O E cutting ρ_π in f. From e and f draw lines $\parallel$ S′ O and $\parallel$ S O to meet in g. g is the point

in (ε) through which ρ_s must lie in order that it along with ρ_ϵ at the chosen distance in front of plane of (ε) may be equivalent to the whole group of locors.

The similar construction for the projection (σ) is also shown in the figure. The distance in front of the plane at which ρ_σ is supposed to act is taken the same as for the other two projections, and this distance is set off in (π) from O S and $\|$ O E, giving the point h on ρ_π, and also in (ε) from O V and $\|$ O E, giving the point i on ρ_ϵ. From i is drawn $ik \|$ E O S′, and along this line from O V is plotted to k the distance of h from O E on the line ih. Then in (σ) the group of locors is equivalent to ρ_e passing through k and ρ_σ at the stated distance from the plane of (σ). **Non-planar and non-parallel locors**

It must be understood that for the solution of any practical problem any *one* of these three representations is sufficient; but to obtain any one of them it is necessary to find all the three, ρ_π, ρ_ϵ, and ρ_σ. In dealing with the balance of engineering structures it is usually convenient to use the construction in plan, and very commonly one particular position of the point d on the line $d\,d^1$ will be found more convenient than others. This method is used in Chapter XII., on Solid Static Structures.

9. In Fig. 57, A E is the resultant. A locor exactly equal and opposite to A E along the same line would balance the others with regard both to vector sum and also to moments. This balancing locor may be called E A to distinguish it from AE. Then we may express the balance by writing, **Balancing locor**

$$AB + BC + CD + DE + EA \;\#\; 0.$$

10. The chain of auxiliary locor lines (P) A B C D E in Fig. 57 may be used as a moment diagram in a similar manner to that already explained for parallel locors. Thus, suppose the sum of the moments round any axis R is wanted. Through R draw a line $\|$ A E $\|$ $a\,e$ and across the space between the lines P E and P A. Let m be the intercept between these **Moment diagram**

lines. Then the triangle whose base is m and whose vertex is I has the same shape as the triangle $e\,a\,p$, whose base is $e\,a$ and whose vertex is p. But the height of vertex above base in the former triangle is the leverage of A E in its moment round R; and the base $a\,e$ is the magnitude of A E. The product of these latter—namely, the moment of A E round R—therefore equals m multiplied by the distance of the pole p from $a\,e$. Call this pole distance from $e\,a$ by the symbol $p\,(a\,e)$. Then,

$$m \times p\,(a\,e) = \text{moment of A E round R,}$$
$$= \text{sum of moments of A B, B C, C D and D E round R.}$$

It is of theoretic importance to recognise this use of the chain (P) A B C, &c., as a moment diagram and the identity of its principle with that of the moment diagram for parallel locors. But the construction is of comparatively little practical utility for two reasons—first, it is as easy to measure directly $a\,e$ and the perpendicular distance of R from A E and to multiply them together as it is to measure m and $p\,(a\,e)$ and to multiply these; and, secondly, in practice we are not at liberty to choose the position of p so as to make $p\,(a\,e)$ an easy number to multiply by and thus to make m measure the desired moment to any desired simple scale. For in practical problems, such as those regarding the forces on bridge, roof, and other work, in dealing with the locors by means of the chain (P) A B C, &c., we find we wish to deal successively with different groups of the locors. For instance, we first wish to find the total moment of A B + B C + C D round any among a given set of axes; then to find the total moment of A B + B C + C D + D E, and so on. Thus, if we chose the position of p so as to give m to read the moments of the first set to an easily readable scale, this position would give a different and most probably an inconvenient scale to which the new m would represent the moments of the other sets.

Moment diagram

The vector diagram $a\,b\,c\,d$, &c., is a rapid and convenient

graphic method of finding the *magnitude* and *direction* of the resultant of any group of the locors. The chain diagram (P) A B C, &c., is an equally simple graphic method of finding the *position* of the resultant. These being found, to calculate the moment of this resultant round any desired axis, the proper procedure is to perform the multiplication of magnitude by leverage directly either by help of measuring these to scale and using ordinary arithmetic, or by help of one of the graphic rules for multiplication already fully explained (see Fig. 8), remembering that the two quantities to be multiplied appear on the paper at right angles to each other and at a distance from each other.

Moment diagram

11. There being only one resultant to a given set of locors, the above construction must give the same line A E whatever chain be used in obtaining it. The chain can be varied in three distinct ways. The directions of its sides are parallel to the radii in the vector diagram from the pole p to the corners $a\ b\ c\ d\ e$. The variability of the directions of the sides of the chain may thus be simply expressed by stating that the pole p may be chosen anywhere.

Variation of moment diagram

12. But for any one position of p, and, therefore, any one set of directions for the chain sides, the decomposition of the first locor A B into A P + P B may be effected at any point of the line of A B. By shifting this point the chain sides are shifted parallelly to themselves through corresponding determinate distances, and the corners of the chain are shifted along the locor lines, some outwards and some inwards, through distances which always maintain constant ratios among themselves whether the displacement along the first line be great or small. If these shiftings of the corners were all outwards or all inwards one might call this kind of variability of the chain that of proportional enlargement or contraction without change of shape; but although these displacements may sometimes be all in one direction, they are not necessarily or generally so. It may, therefore, be referred

Proportional displacement

to as a parallel displacement of the sides or a proportional displacement of the corners. As will afterwards be seen, this proportionality of these displacements is of very important utility in some problems where special difficulties arise. Thirdly, the order in which the locors are taken in this process of addition is indifferent; and the chain is varied by changing this order. However the chain be varied in any one, or in all three of these ways together, the process will give always one and the same straight line A E; that is, the intersection of the first and last link of the chain will always lie in one definite line whose direction is the same as that of the vector sum. As a proposition in pure polar geometry this result is a very interesting one, but engineers will pay more attention to its physical importance. It evidently furnishes an easy means of checking the accuracy of one's work in the summation of locors by this graphic mode.

Proportional displacement

13. In the last section of Chapter VI. the subject of indeterminate abutment thrusts was referred to. When a mass acted on by a given system of loads is kept in balance by two supporting forces acting through two given points, the condition of balance is sufficient to determine three only out of the four elements defining these two forces—viz. two directions and two magnitudes, all the forces being supposed to act in the same plane. If any one element be given, the other three can be at once calculated. Thus if the two forces to be found be called a and b, and if they act through the two known points α and β; then, if the direction of a be given, draw in this direction through α a line to meet the resultant, say ρ, of the known loads as found by the last explained construction. The line joining this intersection with β gives the direction of the second supporting force. The directions of both being now known, their magnitudes are found by drawing in the vector diagram two lines in these directions from the two ends of ρ.

Two supporting forces

If the abutment thrust be indeterminate, the line $r\,\rho$ of Fig. 43 may be found either by assuming the two supporting

forces both parallel to the resultant of the known loads and proceeding precisely as in Chapter VI., Fig. 43, for parallel loads except in using the more general construction now explained for non-parallel loads; or else, the resultant of the known loads being found, *any* direction (such as $e\,\rho$ in Fig. 43) may be chosen as that of *one* of the supporting forces and the other found as above. This gives in the vector diagram an intersection (viz. ρ in Fig. 43) which necessarily lies on the line $\rho\, r$; and through this point a line is to be drawn parallel to the abutment line. This line is the locus of the indeterminate point of intersection of the two supporting forces in the vector diagram.

The former of these two constructions may be conveniently used when the direction of one supporting force is given and when the intersection of its line with that of the resultant of known loads falls outside the limits of the drawing-board. **Two supporting forces**

The line $r\,\rho$ of Fig. 43 being obtained, the correct point ρ upon it can very easily be obtained if only a sufficient extra condition be given regarding the supporting forces. This condition may be given in any of the following forms. The *direction* of one force may be given; the one force being imagined resolved into components in any two rectangular directions, the magnitude of *one* of these components may be given; the ratio of the magnitudes of the components in any definite direction of the two forces may be given, it being necessary in this case to distinguish carefully between a + and a — ratio, the former indicating components in the same direction (and this not always being possible), while the other indicates opposite directions for the two; or again, the magnitude of the abutment thrust may be given, this being merely a special case coming under the second heading.

14. If the given set of loads be balanced by *three* supporting forces acting through three given points α, β, γ; then in order to determine these three there require to be given three conditioning elements besides the conditions of vector **Three supporting forces**

and moment balance. The only case that need be considered here is that in which the *three directions* are given. This case is illustrated in the plan of Fig. 94. Here ρ_π or $D_1 D_2$ is the resultant of the known loads and appears as $d_1 d_2$ in the vector diagram. The directions of 3 4, 4 5, and 5 1 acting through points α, β, and γ are known. 3 4 is produced to its

Three supporting forces

joint μ with ρ_π, and the joint of 4 5 and 5 1 is marked ν. The points $\mu \nu$ are joined and there is drawn $d_1 4^1 \parallel \nu \mu$ and $d_2 4^1 \parallel 34$. Then there is drawn $4^1 5^1 \parallel 45$ and $d_1 5^1 \parallel 1\,5$. This construction evidently gives $d_2 4^1$, $4^1 5^1$, and $5^1 1^1$, the true magnitudes of the three supporting forces at α, β, and γ.

At the end of Chapter XII. this same problem is solved under a set of three conditions much more difficult to deal with than that of three given directions.

15. When a number of parallel locors not in the same plane have to be summed, the method of procedure is as explained in Fig. 58 *b*.

Let the lines of the parallel locors be represented by their point-projections on a plane perpendicular to them. These points are in the figure surrounded by small circles. Through these points draw sets of parallel lines vertically and horizontally. Plot off on a vertical line to a convenient scale the vector magnitudes *ab*, *bc*, *cd*, *de*, *ef*. To the same scale choose

Parallel non-planar locors

a convenient distance from the vector line *af* such as 1, 10, 100, &c., at which to place the pole *p*. Suppose now all the locors to be turned through 90° round a horizontal axis lying on the paper. They will now lie parallel to the paper and their projections will be along the vertical lines A B, B C, C D, D E, E F already drawn parallel to *af*. In the example shown, A B coincides with B C, so that the space B has zero breadth. Form a chain by drawing lines through the successive spaces A, B, C, D, E, and F parallel to the radii of the pencil (p) *a b c d e f*. The line parallel to *pb* is of zero length—i.e. does not need to be drawn, because the space B is of zero width. Let the first and last sides of this chain—namely,

those parallel to $p\,a$ and pf and drawn through the spaces A and F—be produced to meet in I′. Draw a vertical line through I′. Take now a locor of magnitude $a\,f$ perpendicular to the paper (i.e. parallel to the given locors as originally represented on the paper) through any point of the vertical line drawn through I′. This single locor will have the same moment round any axis parallel to the paper and whose projection on the paper is vertical as the sum of the moments of the given locors round the same axis.

Parallel non-planar locors

Again, suppose all the locors turned through 90° round a vertical axis on the paper. Their projections will now be along the horizontal lines already drawn, and will all be perpendicular to $a\,f$. Suppose the pencil (p) $a\,b\,c\,d\,e\,f$ to be turned through 90° so as to bring $a\,f$ horizontal. We could then draw a chain through the spaces A′ B′ C′ D′ E′ F′ lying between the horizontal lines with its sides parallel to the radii of the new pencil. But it is as easy to draw the sides of this second chain perpendicular to the radii of the pencil as it stands in the figure, provided one is furnished with an accurately right-angled set square. There is, therefore, no need to re-draw the pencil. Let the first and last lines of this second chain—namely, those through the spaces A′ and F′—meet in I″. Draw a horizontal line through I″. A locor equal to $a\,f$ taken in the proper sense perpendicular to the paper, and through any point in this horizontal line through I″, will have a moment equal to the sum of the moments of the given locors round any axis parallel to the paper and with a horizontal projection on the paper. Let the vertical through I′ and the horizontal through I″ meet in I. Through I and perpendicular to the paper take a locor equal to $a\,f$. The moment of this round any axis parallel to the paper, whether it be horizontal or vertical, will be the same as the sum of those of the given locors round the same axis. It will, therefore, evidently have a moment equal to the sum of these moments round any axis whatever parallel to the paper; and, therefore, also

round any axis whatever whether parallel or not to the paper, because the components parallel to the axis have no moments. This locor taken through I $\#$ $a f$, the vector sum, is the true locor sum, or resultant as regards moments as well as regards vector sum.

Parallel non-planar locors

Round any axis passing through I the integral moment is zero.[1] A locor through I equal and opposite to $a f$ would balance the given locors as regards moment and vector sum; that is, the system with this balancing vector included would have zero vector sum and have zero moment round every possible axis.

In the moment of a locor we have the product of two directed quantities or vectors whose directions are at right angles to each other. The product is a rotor whose axis is perpendicular to both the factors of the products.

Rotor moment

16. In a rotating body the linear velocity of each point is the product of the angular velocity and of the distance of the point from the axis of rotation, and the direction of the linear velocitity is perpendicular to both the axis of the rotation and the line from point to axis used as a multiplier of the angular velocity. Here we have a directed line—i.e. a vector—multiplied by a rotor whose axis is perpendicular to the vector. The product is a vector at right angles to both the factors. There is an evident and interesting analogy between these results, but the closeness of the analogy must not lead the student to think the two processes identical.

Parallel rotors

17. The addition of two finite rotations round parallel or oblique axes has been already explained (Figs. 53, 54, and 55). That of rotational *velocities* round parallel axes is accomplished graphically in a manner precisely similar to that of the last figure (Fig. 58 *b*). In that figure let A B, B C, C D, D E, and E F represent angular velocities whose axes are parallel,

[1] I is a point in the 'paper,' which may represent *any* plane normal to the given locors. The axis here spoken of may, therefore, be otherwise described as any axis passing through any point of the line parallel to the given locors through I.

having their projections at the points surrounded by the circles, and whose magnitudes are represented to scale by *a b*, *b c*, *c d*, &c. The construction is carried out precisely as before. There is obtained the resultant axis I, and the magnitude of the resultant angular velocity is *a f*.

Parallel rotor moment diagram

18. In Fig. 59 let I be the same point as I in Fig. 58, and through it let there be drawn two straight lines parallel to *p a* and *p f*. Let 0 be any point, and through it draw a line $\parallel a f$. Let the intercept on this line between the last two lines be v. Then, as before, $v \times \overline{p\,(a f)} = a f \times$ perpendicular distance of I from line through 0

$$= a f \times \overline{\mathrm{I}\,(v)}.$$

Now, *a f* being the angular velocity of a body rotating round axis I, the linear velocity of the point 0 resolved in the direction of v is $a f \times \overline{\mathrm{I}\,(v)}$. Thus v represents, to a certain scale dependent on the pole distance $\overline{p\,(a f)}$, the vertical component of the linear velocity of any point 0. The two lines drawn through I, therefore, form a complete diagram of the vertical components of velocity of each and every point of the body rotating round I. The scale of the linear velocities will be a convenient one if the pole distance $p\,(a f)$ be chosen a simple one to multiply by, for instance, 1, 10, 100, or 1,000 radians per second or per minute.

Linear velocity diagram of rotating body

Now, through the same point I let two lines be drawn perpendicular to *p a* and *p f*, and let h be the intercept between them of a horizontal line drawn through 0. By reasoning similar to the above it is evident that, to the same scale as that to which v represents the vertical component of linear velocity, h represents the horizontal component of this linear velocity of point 0 of the rotating body.

These two pairs of lines perpendicular to each other and jointed in I thus form a complete diagram of the linear velocities of all points of the body rotating round I with angular velocity *a f*. To complete the representation a circular arrow

should be drawn round I to show without ambiguity the direction of the rotation. This arrow will also show distinctly the directions of all possible vertical and horizontal components of linear velocity of all points. The total linear velocity $= \sqrt{v^2 + h^2}$, which can easily be found graphically after v and h have been obtained on the diagram. In place of the circular arrow, four straight arrows may be drawn in the four angles in which the intercepts in the directions of the arrows measure the linear velocities.

Parallel rotor moment diagram

We have here spoken of Fig. 59 as giving the linear velocities due to a single angular velocity round I, assuming from the proof given on p. 72 that for each point the resultant of the linear velocities due to the separate component angular velocities round different axes is equal to the linear velocity due to the single resultant angular velocity round the resultant axis I. This, however, can easily be proved directly and in detail by help of Fig. 58, proceeding precisely by the method of Fig. 41.

Moment diagram of non-parallel rotors

19. When two angular velocities are about axes which intersect, it has already been shown that their sum is to be found by representing their magnitudes by proportional lengths along their axes and, treating these representations as locors, finding their sum as if they were locors. It follows that any number of co-planar instantaneous rotors, whether parallel or not, can be summed up by exactly the same graphic constructions (see Figs. 41 and 57) as have been given for co-planar locors. The diagrams drawn form diagrams of linear velocities if the rotors be angular velocities. When the rotors are neither co-planar nor parallel, they must be resolved into two sets, one all lying in any chosen plane and the other all perpendicular to this plane and therefore parallel to each other. These two sets are to be summed up separately by the methods already explained. If the two resultant axes intersect, the two resultant rotors can be combined into a single rotor. If not, the system reduces either to a pair of rotors or else to a single rotor combined with a locor.

20. Angular momenta are simply angular velocities multiplied by masses. They form rotors which are to be dealt with precisely as angular velocities by the foregoing methods. The product of an angular momentum and a radius perpendicular to it is evidently a linear momentum.

These are the only rotors whose moments have to be investigated in ordinary mechanics.

Moments of angular momenta

The product of two rotors with parallel axes is generally an undirected quantity. For instance, that of an angular velocity and of a force-moment is a rate of doing work (horse-power). That of an angular velocity and the moment of an acceleration of linear momentum is a rate of increase of kinetic energy. That of an angular rotation and of the moment of an acceleration of momentum is a quantity of kinetic energy. That of an angular rotation and of a force-moment is a quantity of mechanical work done.

21. Suppose we have two locors along parallel but different lines, oppositely directed and equal in magnitude. The construction given above in Fig. 57 fails in this case. In Fig. 60 let A B, B C, $a\,b$, $b\,c$ represent the two locors. Choose any pole p and draw A P $\parallel a\,p$, and B P $\parallel b\,p$, and C P $\parallel c\,p \parallel a\,p$. The lines A P and C P being parallel, do not intersect, and it becomes impossible to find a position for the locor sum or resultant. Evidently, however, the two final lines P A, P C still form the correct moment diagram for the pair of locors—that is, the moment of the pair round any axis whatever is the intercept between these two parallel lines on the line drawn through the axis parallel to $a\,b$; the scale being determined by the pole distance $p\,(a\,b)$. This moment has the same magnitude round all possible axes perpendicular to the plane containing the two locors. The resultant vector being zero, it has no direction. The system has zero vector magnitude and has lost the two other characteristics of locors—viz. direction and position. It retains, however, its character of giving a moment, this moment being a rotor and having a definite

Locor and rotor couples

direction for its axis. The resultant of the two locors cannot be said to be a rotor, but it has lost all locor character except this of giving a moment which is a rotor.

When the two equal and opposite locors are forces, the pair is called a 'couple' or 'force-couple.' If the two equal and opposite forces be A B and C D, the couple is written (A B $\nparallel$ C D).

Locor and rotor couples

Again, if in the above figure A B, *a b*, and B C, *b c*, represent equal and opposite angular velocities round parallel axes, the magnitude of the resultant angular velocity is zero and its axis has neither position nor direction. The two parallel lines P A and P C, however, continue to form a diagram of linear velocities of all points of the body that has these two simultaneous angular velocities superimposed upon it. These linear velocities are all equal—that is, the body is moving with a velocity of translation only. The angular velocity has become infinitely small and the distance of the axis of rotation infinitely great; the product of the two, or the linear velocity, being in the limit the distance between the parallels A P and C P taken $\parallel$ *a b*.

Here the combination of two rotors has lost all rotor character except that of giving a moment which is a vector. It cannot, however, be correctly said that the resultant of the two angular velocities is a translatory linear velocity, this latter really being the moment of the resultant zero angular velocity. The resultant of the two angular velocities may be called a 'rotor-couple.'

Equivalent couples

22. Neither a locor-couple nor a rotor-couple having any but moment properties, all couples giving the same moment must be considered as equivalent; any one of an equivalent set of couples may be substituted for any other so far as vector or moment phenomena are concerned. Thus for the two equal and opposite forces at a given distance apart may be substituted two other equal and opposite forces of different magnitudes and directions from those of the original couple, provided

the distance apart of the new pair is greater or less than that of the first pair in the inverse proportion in which the magnitudes are less or greater, and provided also that the plane of the second is parallel to the plane of the first pair. This latter condition ensures that the axes of the two couples have the same direction. A similar substitution may be made in the case of rotor couples.

When students observe how freely this substitution is made use of in text-books of mechanics they are apt to imagine that the effects of these equivalent couples as applied to pieces of material are in every respect the same. They should carefully avoid this error. The distributions of stress and strain throughout a piece of material are among the most important conditions that engineers have to inquire into. The states of stress and strain in bodies due to the action of different but 'equivalent' couples are wholly different. If the question to be investigated involves stress and strain, the above sort of substitution cannot be made. If only the bending moment on a certain section of material has to be found, then this substitution is permissible; but no problem of strength or stiffness is completely solved by the calculation of bending moments alone, and this—viz. the case of bending moments—is the only item of stress and strain calculation in which an 'equivalent' couple may be substituted for the actually existing couple. A similar warning may be given regarding the substitution of a 'resultant' of any kind such as are explained in this chapter. The resultant can only be substituted for the group of units for purposes relating to the 'vector sum' or to the moments. The resultant applied instead of the actual forces would not have even approximately the same effect in stressing and straining the material.

Equivalent couples

Resolution and composition of couples

23. In Fig. 61 let the plane of a couple be perpendicular to the paper; let its forces be represented by $a\,b$ and $b\,a$; and let its moment be represented to a convenient scale by $a\,m$. $a\,m$ may be taken as the axis of the couple; it is parallel to the

Resolution and composition of couples

paper, and perpendicular to the plane of $(a\,b ⋕ b\,a)$. Each of these two equal forces may be resolved in any two rectangular (or oblique) directions $a\,c$ and $c\,b$. Then $(a\,c ⋕ c\,a)$ will be equal and opposite forces, and will form a couple whose moment is $a\,n$ obtained by drawing $a\,n \perp a\,c$ and $m\,n \perp c\,b$. Also $(c\,b ⋕ b\,c)$ are equal and opposite forces forming a couple whose moment is measured by $n\,m$. Again, $a\,n$ and $m\,n$ are perpendicular to the planes of these two couples $(a\,c ⋕ c\,a)$ and $(c\,b ⋕ b\,c)$, and these may, therefore, be taken as their axes. It is now evident that couples may be resolved and compounded, subtracted and added in exactly the same way as vectors or instantaneous rotors, and that these operations may be effected by treating as vectors or instantaneous rotors lines measured along their axes (such as $a\,m$, $a\,n$, $n\,m$) to such lengths as will represent the moment magnitudes to any convenient scale. These axial lengths are complete graphic descriptions of the couples, and the term 'axis of a couple' is usually employed to indicate this definite length perpendicular to the plane of the couple.

24. A locor-couple combined with a single locor R perpendicular to the axis of the couple evidently reduces to a single locor, which is equal and parallel to R but along a different line. If a set of locors reduce to a couple and a single locor parallel to the axis of the couple, these cannot be added so as to give any simpler result.

General locor and rotor reduction

If the set reduce to a couple and a single locor R oblique to the couple axis, the couple may be resolved into two components, the axis of one component being parallel and that of the other perpendicular to R. The latter component combines with R to give a single locor ⋕ R, and the whole set is now reduced to the above last-mentioned case.

If the set reduce to two non-parallel and non-intersecting locors, Q and R, either of these, say Q, may be resolved into two components, one component being equal and opposite to R (i.e. ⋕ R). This component forms with R a couple, and

the whole set is now reduced to a couple and a single locor, which is in general oblique to the axis of the couple, but by the previous method can be further reduced to a couple and a locor parallel to the couple axis.

It is, however, questionable whether there are many cases in which there is any practical advantage to be gained by substituting a couple and locor (which together really mean three locors) for a system of two single non-intersecting locors. Two oblique non-intersecting locors may always be reduced to two other perpendicular non-intersecting locors. Thus, in Fig. 62, let *a b* be the one locor in the plane of the paper and *c d* be the projection on the same plane of the other. This latter can be resolved at the point *c* where it cuts the plane of the paper into two components, of which the projection *c d* is one and the other is perpendicular to this plane. *c d* and *a b* can be added, the sum being a locor *e f* in the plane of the paper. The system is now reduced to *e f* in this plane and the vertical component of the second locor perpendicular to same plane and passing through *c*. One is at liberty to choose any plane that may be most convenient for this reduction, and for this reason among others this solution may in many instances be more convenient and useful than the reduction to a couple and a perpendicular locor, for which latter there is only one definite plane possible for the couple.

General locor and rotor reduction

CHAPTER IX.

KINEMATICS OF MECHANISMS.

1. In this chapter there are dealt with the relations between the velocities and accelerations of velocity of the various parts of a Mechanism.

A 'mechanism' may be defined as a combination of frames, plates, bars, or flexible members jointed together in such manner as to ensure that, while the parts may move relatively to each other, the relative positions of all parts are determinate for each given possible relative position of any two parts.

Definition of 'mechanism'

The different rigid members of a mechanism may be termed 'links' or 'bars.' The flexible members may be called 'bands' or 'belts.'

Bed-plate

That bar relatively to which the motions of the other parts are measured is here termed the 'base-plate' or 'bed-plate' or 'frame.' The motions of these other parts are thus considered as taking place in the 'field of the bed-plate;' and in this field the parts of the bed-plate itself have, of course, no displacement, velocity, or acceleration of velocity.

2. In the next chapter (Chapter X.) there is explained the relation that must subsist between the number of joints and that of links in order that a flat structure may be 'stiff'—i.e. non-deformable without stretching, contracting, or bending the links. The relation is

Joints and bars

$$l = 2j - 3,$$

where l = number of links and j = number of joints. When a link has more than two joints in it, only two of them must be included in the count of j; and such a link, which is called a beam-link, is to count as one only.

If a flat pin-and-eye linkage have a number of links *less* than the above, it is no longer stiff and can be moved into different shapes. If the number of links be *one* less than the above, or

$$l = 2j - 4,$$

there is *one* degree of freedom in the possible motion. This means that, although there is an infinite number of possible shapes, still for each definite position of two parts relatively to each other there is a definite relative position of every other pair of parts. This agrees with our definition of a mechanism. The definition of a *machine* will be given in Chapter XIII. A machine has two more bars than a mechanism with the same number of joints, and two of its bars are not rigid, but are capable of resistant deformation. In a pin-and-eye joint mechanism the relation between the numbers of links and of joints is, therefore,

$$l = 2j - 4.$$

Joints and bars

In a quadrilateral, for instance (see Fig. 67), which is the simplest mechanism, there are 4 joints and 4 links, and

$$4 = 2 \times 4 - 4.$$

In applying this rule only two joints are to be counted to each bar, no matter how many may actually exist in it. Thus in Fig. 70 there are 6 bars; there are also 7 joints, but two of the bars have in each 3 joints, so that the number of joints to be counted in order to apply the criterion is 5. Now

$$6 = 2 \times 5 - 4,$$

and the mechanism has, therefore, definite motions.

A pin-and-eye, or full and hollow cylindric, joint is called by Reuleaux a 'lower pair,' as distinguished from the joint between two spur-wheels in gear with each other, which latter is called a 'higher pair.' The above criterion-equation requires modification for mechanisms including higher pairs. For each 'higher pair' joint existing in the mechanism, the number of bars becomes *one more* than that given by the

above rule. This is evident because a higher-pair joint may be looked upon as a substitute for *two* lower-pair joints with a link between them. The suppression of one of these joints would correspond to the suppression of two links, but actually only one link disappears in the conversion of the link and two joints into a single higher-pair joint.

Rigid-bar mechanisms

3. The motion of any one point in a bar relatively to any other point in the same bar may be conveniently resolved into two components: one along the line joining the two points, which will either lengthen or shorten the dimensions of the bar along this line; the other at right angles to this same line. If the bar be 'rigid,' that is, if its dimensions be incapable of alteration, the former component is always zero. Rigidity does not interfere with the latter component taking any value whatever, great or small. Thus, in a rigid bar, the instantaneous displacements and velocities of the various pairs of points relative to each other are all *normal* to the lines joining the pairs of points, and all parts have the same angular velocity.

Motion-paths

4. The graphic determination of the simultaneous positions of the various bars of ordinary mechanisms is easy. When difficulties occur, as in some reversing link-motions, the methods here explained enable them to be readily overcome. The *loci* of the successive positions of the various parts may be called the 'motion curves,' or more simply the 'paths,' of these parts. These paths are drawn in on the 'mechanism diagram.'

Displacement diagrams

5. From these 'paths' the *displacements* from any assumed initial configuration can be directly measured. It may, however, be often advantageous to have a separate 'displacement diagram,' consisting of a series of curves showing the successive simultaneous displacements of all important points of the mechanism as vector-radii from one and the same pole. These curves in the 'displacement diagram' are, of course, exact copies of the 'paths' in the 'mechanism diagram.'

Let ABC and BDE in Fig. 63 be two bars jointed together at B. Let P′ be the pole of the displacement dia-

gram, and let the curves A′A′, B′B′, D′D′ be the displacement curves of the three points A B D. All these curves, of course, pass through the pole P′. P′A′$_1$, P′B′$_1$, and P′D′$_1$, being simultaneous displacements of A B and D, draw on A′$_1$B′$_1$ and B′$_1$D′$_1$ as bases, the triangles A′$_1$B′$_1$C′$_1$ and B′$_1$D′$_1$E′$_1$ similar to the triangles A B C and B D E in the mechanism. It can easily be shown that P′C′$_1$ and P′E′$_1$ are the simultaneous displacements of C and E if the two bars be rigid. By joining all the points C′ and E′ found by such constructions, the displacement curves of C and E for a rigid-bar mechanism can be drawn in. Numerous simultaneous points on the various displacement curves should be marked, and numbered 1, 2, 3, 4, &c.

The advantage of such a displacement diagram over the set of 'paths' dispersed over the mechanism diagram consists in the greater facility of comparison between the displacements of the various parts of the mechanism that it affords. Thus, comparing simultaneous points A′ and C′ belonging to the same bar, the vector A′C′ is the displacement of C past (or relatively to) A in the base-plate field. Conversely, C′A′ is the displacement of A past C in the same field. The same holds for points in different bars; thus, D′C′ is the displacement of C past D in the base-plate field. **Displacement diagrams**

It may be noted that any line, such as A′C′, belonging to *one* bar is perpendicular to the line bisecting the angle between the simultaneous and 'initial' positions of the line A C in the mechanism. This does not, however, apply to a line, such as E′C′, joining points belonging to *different* bars. This latter vector may be looked on as the sum (E′B′ + B′C′), to each of which components the above rule applies, although it does not apply to the resultant.

6. The method of obtaining the velocities by taking the small differences of the displacements, which method is the basis of kinematic analysis developed by means of the differential calculus, has often been adopted as a graphic process for **Differential velocity diagrams**

the solution of specially complicated problems. After constructing the velocity hodographs, the same method may be followed to find the velocity accelerations. As a graphic process, however, this method is capable of no accuracy; it s, in fact, practically useless.

Reuleaux's centroids

Professor Reuleaux's method of centroids, more properly called axoids, has now become famous. These axoids are, however, very tedious of construction, and when constructed furnish no direct easy means of measurement of useful quantities.

7. The method now given furnishes velocity and acceleration diagrams, somewhat similar in appearance to stress diagrams, which show the true directions and magnitudes to scale of the velocities and velocity accelerations of all points in the mechanism; there being one pole only for each diagram from which all vectors radiate, so that the velocities or accelerations of all parts and at all times of the complete cyclic period can be compared with maximum facility.

Polar velocity diagram

Fig. 64.—Let A B C D be a rigid bar. Suppose the velocity of A over the base-plate P to be known. Choose any pole p, and draw $p\,a$ parallel to the velocity of A, and of a length to represent its magnitude to any scale considered convenient for the velocity diagram. If now the angular velocity ω of the bar be also known, $a\,b$ may be plotted perpendicular to A B, and equal in length to $\omega \cdot \overline{AB}$ to the above velocity scale. Then, $p\,b$ is the velocity of B over the base-plate. If, instead of ω being known, we know the direction of the velocity of B, then $p\,b$ may be drawn from p in this known direction to intersect in b the line $a\,b$ drawn $\perp$ A B from the point a. This gives $p\,b$ the velocity of B; and the angular velocity of the bar may, if desired, be calculated by dividing $a\,b$ by A B. Since the (relative) velocity of C round A is perpendicular to A C, and its relative velocity round B is perpendicular to B C; if $a\,c$ and $b\,c$ are drawn perpendicular to A C and B C, their intersection c gives $p\,c$ the velocity of C through the base-plate field. Similarly, $p\,d$ is found to measure the velocity of

D. The diagram gives not only the velocities over the base-plate P, but also all the velocities of pairs of points relative to each other. For instance, *b d* is that of D round B, and *d b* is that of B round D, these relative velocities being through the field of the base-plate P.

It is clear that the figure *b a c d* forms a diagram of the bar B A C D to a diminished scale and turned through a right angle in the direction of ω. It may be called the 'velocity image' of the bar. Further, on this new diagram of the bar, altered in scale and rotated through 90°, the pole *p* represents the position of the instantaneous axis of rotation. Theoretically, the original diagram A B C D, with the position P of the instantaneous axis added, would serve equally well as a diagram of velocities, the scale being chosen suitably so that P A would represent the velocity of A. In fact, this construction is really the gist and basis of Reuleaux's method above mentioned, various modes of finding the instantaneous axis being adopted according to circumstances. But for practical graphic construction it cannot be so used, for several reasons. First, the usual variation of the position of the instantaneous axis is extremely inconvenient, and in almost all mechanisms this axis periodically recedes to an infinite distance. Secondly, the scale to which the diagram P A B C D could represent the velocities is always varying throughout the periodic motion of the mechanism; it is always *necessarily* an awkward scale to measure to, and it periodically becomes in most cases an impossible scale by becoming infinitely large. Thirdly, the various bars of a mechanism have all different instantaneous axes, and the scales of the velocities would be entirely different for the different bars. As will be shown presently, in the method explained in this chapter the velocity diagrams of all the bars of even the most complicated mechanisms are grouped together so as all to radiate from one pole, and so as to be to the same scale for all the bars and at all times throughout the periodic motion of the mechanism.

Velocity image

8. A similar construction is applicable to accelerations of velocity. In Fig. 65 let A B C D be one rigid bar. Let the acceleration of velocity of point A through the field of the base-plate be known, and represented in direction and magnitude by $p' a'$ drawn from any convenient pole p' to any convenient acceleration scale. The acceleration of B can be obtained by adding to the vector $p' a'$ the vector acceleration of B in its relative motion round A. If ω be the angular velocity of the bar in the base-plate field, and if ω' be the acceleration of angular velocity, the radial or centripetal component of velocity acceleration is $\omega^2 \cdot \overline{AB}$ and the tangential component is $\omega' \cdot \overline{AB}$. Thus plot $A\gamma = 1$, and from γ plot backwards towards A the magnitude ω^2 to any convenient scale. From the point so obtained plot to same scale $\omega' \perp AB$ to the point α. From B draw $B\beta \parallel \gamma\alpha$ to meet $A\alpha$ in β. Then $B\beta$ equals the acceleration of the velocity of B round A. The whole acceleration of relative velocity is, therefore, $\overline{AB} \cdot \sqrt{\omega^4 + \omega'^2}$, and its direction is inclined to B A by the angle $\tan^{-1} \frac{\omega'}{\omega^2}$.

Accelera-tion diagram

This angle is the same for every pair of points in the same rigid bar, for every part of the bar has at each instant the same ω, and, therefore, also the same ω'; and, since the magnitude of the acceleration of one point round any other is proportionate to the distance between them (e.g. that of B round A is $\propto$ A B), therefore, if $a' b'$ be drawn inclined to B A at the angle $\tan^{-1} \frac{\omega'}{\omega^2}$ and of length $AB\sqrt{\omega^4 + \omega'^2}$, and if the figure $a' b' c' d'$ be made similar to that of the bar A B C D, then $p' b'$, $p' c'$, $p' d'$, will be the accelerations of the points B C D in the base-plate field. Further, $a' c'$, for instance, is the acceleration of C round A. In the graphic construction it is simplest to plot $a' \beta' = \overline{AB} \cdot \omega^2$, and parallel to B A (not A B), and $\beta' b' = \overline{AB} \cdot \omega'$ perpendicular to A B and in the direction given by the sign of ω'. The radial component is usually obtainable from the already constructed velocity diagram,

where the velocity of B round A is called $a\,b$, and the radial acceleration is therefore $\frac{(a\,b)^2}{A\,B}$. Fig. 66 gives the two most ready graphic constructions for calculating $\frac{(a\,b)^2}{A\,B}$. In Figs. (1) and (2) the velocity $a\,b$ is plotted along B A from B to β, towards A in (1), and away from A in (2). From B as centre with $B\,\beta = a\,b$ as radius, a circular arc is struck intersecting in β_1 any other radius from A. From β is drawn $\beta\,\beta'$, parallel to that other radius, and intersecting $B\,\beta_1$ in β'. Then $B\,\beta'$ is the radial acceleration $\frac{(a\,b)^2}{A\,B}$. In (3), $(a\,b)$ is plotted from B as $B\,\beta$ perpendicular to A B, and a circular arc with centre in B A is struck through A and β. This arc cuts the diameter A B in β' giving B β' the desired radial acceleration.

Acceleration diagram

9. If the bar be plotted to the scale $m'' = 1$ inch, m being a fraction; and if the velocity be plotted to the scale $n'' = 1$ inch per second; then such constructions give the acceleration to the scale $\left(\frac{n^2}{m}\right)$ inch $= 1$ inch per second per second. If it be desired to *diminish* the size of the acceleration diagram from this scale in the ratio q, it is only necessary to make the constructions of Fig. 66 with the radius A B *increased* in the same ratio q. To increase the size of the acceleration diagram, in any ratio, diminish A B in Fig. 66 in the same ratio.

Scale of acceleration diagram

10. It is evident that the figure $a'\,b'\,c'\,d'$ of the acceleration diagram is simply a reproduction of the figure A B C D of the bar altered in scale and rotated through an angle $\left\{180° - \tan^{-1}\frac{\omega'}{\omega^2}\right\}$ in the direction of ω, where, in $\tan^{-1}\frac{\omega'}{\omega^2}$, the sign of ω' is to be taken positive or negative according as it is in the same or the opposite direction to that of ω. The figure $a'b'c'd'$ may be called the 'acceleration image' of A B C D. In this image p' is the point of the bar if the bar extends so far, or of its field if it does not extend so far, which suffers no

Acceleration image

acceleration or is moving uniformly in a straight line. This point does not in general coincide with the instantaneous axis of rotation. If the velocity diagram were rotated and altered in scale, and placed on top of the acceleration diagram so that $a' b' c' d'$ coincided with $a b c d$, then $p' p$ would represent in direction and magnitude the acceleration of that line in the bar or in the field of the bar which is at any time the instantaneous axis.

Acceleration image

Both the velocity image and the acceleration image of a rigid bar moving without rotation, i.e. with a motion of translation only, reduce to a single point. The smaller the rotational velocity the smaller will these images be.

Integral momentum. Acceleration of momentum

11. If G be the centre of inertia of the bar and the similar points g and g' be plotted in the velocity and acceleration diagrams, then the products of the bar-mass by $p g$ and $p' g'$ are respectively the integral momentum and the integral acceleration of momentum of the whole bar.

12. In what follows the capital P will denote the base-plate through whose field the velocities, &c., are reckoned.

The pole of the velocity diagram will be called p. The pole of the acceleration diagram will be called p'.

Points in the mechanism will be named by capital letters, A B C, &c.

Nomenclature

The corresponding points in the velocity diagram will be named by the same letters in small type, $a b c$, &c.; so that $p c$ will denote the velocity of C over the base-plate and $b c$ the velocity of C round B, and $c b$ that of B round C.

The corresponding points in the acceleration diagram will be named by accented small letters; this being in accordance with the common mathematical convention, whereby x' represents $\frac{d x}{d t}$.

In finding, for instance, the point b, it is sometimes necessary to find other points which are not afterwards required in the completed diagram. When such construction points

need to be named, they will be called β_1 β_2, &c., if used to find b in the velocity diagram, and β'_1 β'_2, &c., if used to find b' in the acceleration diagram. **Nomenclature**

In the displacement diagrams described above, the accented capitals A′ B′ C′, &c., are suitable.

13. The simplest mechanism is that with four rigid bars and with two joints, P_1 P_2, in the base-plate, and two joints A B coupling the other three bars together. An example is shown in Fig. 67, the calculations being made for five different phases of the periodic motion.

The velocity of the crank-pin A is supposed known at each phase. From any pole p, and to any convenient scale, this velocity $p\,a$ is plotted perpendicularly to $P_1 A$. From p a line is drawn perpendicularly to $P_2 B$. Evidently the extremity b of $p\,b$, the velocity of B, must lie in this line. But also $p\,b \#$ $p\,a$ plus a velocity perpendicular to A B. Therefore, from a a line is drawn perpendicular to A B to meet the above line in b. Thus $p\,b$ is determined. In the example $p\,a$ is taken of the same magnitude at all the five phases.

To obtain the acceleration diagram we assume the acceleration of A. On the supposition that $p\,a$ is constant in magnitude, this acceleration is also constant in magnitude $=\frac{(p\,a)^2}{P_1 A}$, and is wholly radial. **Four-bars**

From any pole p' this acceleration $\frac{(p\,a)^2}{P_1 A} = p'a'$ is plotted parallel to $A P_1$ (not to $P_1 A$).

The calculation of the magnitude is performed by the graphic construction previously explained. By the same construction the magnitudes $\frac{(p\,b)^2}{P_2 B}$ and $\frac{(a\,b)^2}{A B}$ of the radial components of the accelerations of B round P_2 and round A are found and plotted off from p' (as $p'\beta'$) and from a' (as $a'\beta'$) parallel to $B P_2$ and to B A. From these two points β' thus obtained, lines are drawn perpendicular to $B P_2$ and to B A. The point b' sought for must lie on both of these last lines,

and is, therefore, at their intersection. The acceleration $p' b'$ of the joint B through the field of P is thus obtained for the five different phases of the motion. The method of procedure is plain. Each joint of the mechanism is a point in two different bars, and therefore the calculation for that joint may be approached, as it were, from two different sides. In each of the two calculations there is one element missing, and the last stage of the calculation cannot be completed directly; for example, approaching the calculation of the acceleration of B through A, we can calculate the radial component (parallel to B A) of the acceleration that has to be added to that of A; while of the tangential component the direction only is known, but this gives a line in which the desired point must lie. Another conditioning line being similarly found by approaching the calculation in another way, the point is found at the intersection of these two lines.

Four-bars

Steam-engine mechanism

14. In the ordinary steam-engine with guide bars, the radius bar B P_2 swinging in the base-plate bearing at P_2 is replaced by the cross-head sliding in straight guides which form part of the base-plate. The effect is the same as if B P_2 were infinitely long. On account of the cross-head joint being guided in a straight line passing through the crank journal centre, a symmetry is given to the motion which materially lightens the labour of drawing complete velocity and acceleration diagrams. Fig. 68 illustrates this. The angular velocity of the crank is taken as uniform. Therefore, the linear velocity of A is uniform and the points a all lie in one circle whose centre is p.

Here the four positions 1, 2, 3, and 4 of the crank-pin A are taken equidistant from the dead-points O and O′. Therefore, the two cross-head positions B_1 and B_4 coincide, as do also B_2 and B_3. Therefore also the four velocities $p\,a_1$, $p\,a_2$, $p\,a_3$, and $p\,a_4$ are equally inclined to the velocity line $p\,b$, and the four points a_1, a_2, a_2, a_4 are equidistant from the line $p\,b$. Also at 1 and 2 the connecting rod has the same inclination

to the centre line, which inclination is equal and opposite to that at 3 and 4. Thus the lines $a_1 b_1$, $a_2 b_2$, $a_3 b_3$, and $a_4 b_4$ are equally inclined to $p b$, the inclinations of the former two being opposite to those of the latter pair; and, therefore, the velocities $p b_4$ and $p b_1$ have equal magnitudes, as also have $p b_3$ and $p b_2$. Therefore also the radial accelerations $a'_1 \beta'_1$, $a'_4 \beta'_4$ have equal magnitudes, as also $a'_2 \beta'_2$ and $a'_3 \beta'_3$, and are equally inclined to $p' b'$ on opposite sides; while also the tangential accelerations, $\beta'_1 b'_1$, &c., are equally inclined to the same line. Therefore, finally, b'_1 coincides with b'_4, and b'_2 with b'_3, but $p' b'_2 = p' b'_3$ differs in magnitude as well as direction from $p' b'_4 = p' b'_1$.

Steam-engine mechanism

This symmetry is, of course, destroyed by want of uniformity in the rotation of the crank.

The joint lines of the bars of a mechanism, the velocity lines, and the acceleration lines need be drawn in full for one position only. The results for the other positions are indicated by numbered points on the three sets of curves, which are the *loci* of the corresponding points or extremities of lines. The first set of curves are the paths of motion of the joints. The second series of curves are the hodographs of the velocities of these same joints. The third series are the *loci* of the extremities of the lines representing the velocity accelerations. In Fig. 68 the velocity and acceleration diagrams marked (C) (D) are constructed in this way, and show the velocities and accelerations of the joints A and B for thirty-two equidistant positions of the crank-pin.

15. Six-bar motion is nearly equally easy to deal with by positions this method.

Six-bars

The first example given in Fig. 69 is quite simple, because the velocity $p a$ of the joint A is assumed as known, the bar P_1 A being one of the quadrilateral P_1 A B P_2. The determination of the velocity $p b$ is, therefore, the same as that given already. Thus $p b$ and $a b$ are drawn perpendicular to P_2 B and A B, and their intersection gives b. Then the triangle

abc is made similar to ABC. pd is then drawn perpendicular to P_3D, and cd to CD, the intersection giving pd the velocity of D. To find the velocity of E, there are drawn pe and de perpendicular to P_3E and DE. The construction of the acceleration diagram here offers no special difficulty.

16. The solution of the next example in Fig. 70 is not quite so direct, because here the velocity assumed as known, namely, pa that of A, is that of a joint in the *pentagon* P_1ABCP_2. First, pa is drawn of the known magnitude and perpendicular to P_1A; and then $a\beta$ of indefinite length perpendicular to AB. Then $p\delta$ and pc are drawn of indefinite length perpendicular to P_3D and P_2C—that is, in the directions of the velocities of D and C. The points bc and d now sought for are known to lie on the lines $a\beta$, pc, and $p\delta$, and also it is known that the line joining b and d is perpendicular to BD. Any point β on $a\beta$ is chosen, and from it $\beta\delta$ drawn perpendicular to BD; and then the triangle $\beta\delta\gamma$ is constructed similar to BDC, corresponding sides being perpendicular. The triangle bdc that is sought for must evidently be similarly placed to $\beta\delta\gamma$ between the lines $p\delta$ and $a\beta$. Otherwise stated, the locus of γ for various positions of β and δ is a straight line passing through the intersection of the line $a\beta$ and $p\delta$. Therefore, γ is joined with the intersection of $p\delta$ and $a\beta$, and this line is produced to intersect pc in c. This gives the true position of c, and the triangle dcb is then completed by drawing cd and cb perpendicular to CD and CB to meet $p\delta$ and $a\beta$. The point e is obtained by drawing pe and de perpendicular to P_3E and DE.

Six-bars

The acceleration diagram has in this case to be constructed according to a similar indirect method. The acceleration of A being supposed known can be plotted at once. Then the radial components of the accelerations of B round A, of C round P_2, and of D round P_3, are calculated and plotted off from a' and p' in their proper directions—namely, parallel to

B A, C P_2, and D P_3; their magnitudes being $\frac{(b\,a)^2}{\mathrm{BA}}$, $\frac{(p\,c)^2}{\mathrm{P_2C}}$, $\frac{(p\,d)^2}{\mathrm{P_3D}}$. From the three points so obtained, three lines, which we may call β', γ', and δ', are drawn of indefinite length perpendicular to B A, P_2 C, and P_3 D. The tangential components of the above three accelerations lie along these lines, which, therefore, contain the three points b', c', and d' sought for. On the line δ' any two points, δ'_1 and δ'_2, are chosen, and from each the centripetal acceleration $\frac{(c\,d)^2}{\mathrm{CD}}$ of C round D is plotted parallel to C D; and from the two points thus obtained are drawn two lines perpendicular to C D to meet the line γ' in two points, say γ_1' and γ_2'. On the two bases $\delta_1'\,\gamma_1'$ and $\delta_2'\,\gamma_2'$ are constructed two triangles similar to D C B, whose two vertices may be called β_1' and β_2'. Neither of these points β_1' β_2' will be found to lie on the line β', and their distances from this line may be taken as measures of the **Six-bars** errors involved in the two guesses δ_1' δ_2' at the position of d'. The error thus found in the resulting position of β' is a linear function of the error in the guessed position of δ'; and, therefore, the interpolation between these two errors in order to reduce them to zero is to be performed by simple proportion. This linear interpolation is at once effected graphically by drawing a line through β_1' and β_2', and producing it until it meets the line β'. The intersection thus found will be the true position of b'. Or, otherwise, the two error-distances of β_1' and β_2' from the line β' may be plotted off from the points δ_1' and δ_2' perpendicularly to the line δ' (or both in *any one* direction inclined to δ'), and through the two points thus obtained a straight line is drawn to cut the line δ' in the true position of d'. From d' or b' thus determined, the other points are constructed as usual. This construction of the acceleration diagram of Fig. 70 will be easily followed without further explanation. From b', found as above, is plotted off $\frac{(c\,b)^2}{\mathrm{CB}}$

parallel to C B, and from the point so obtained is drawn a line perpendicular to C B to meet the line γ' in c'. Then on $b' c'$ as base is constructed a triangle similar to B C D, the other corner of which is marked d' and should fall on the line δ' if the drawing has been accurate.

Six-bars

This indirect method of 'two trials and two errors' and linear interpolation between them is adopted in drawing the velocity and acceleration diagrams for the ordinary forms of steam-engine reversing link motion. These diagrams could not be obtained except by this method.

17. In the common steam-engine mechanism we have already had a case of one bar *sliding* on another—namely, the cross-head sliding in the guide-bars of the bed-plate. A circular slot in which sliding takes place may, of course, be looked upon as simply an incomplete pin-joint of large size, the radius of the pin becoming infinite when the slot is straight. But when the radius of the slot is large, this manner of regarding the joint is not practically useful owing to the very distant position of the centre. A more direct application of the present graphic method to sliding joints is effected thus: If B be a bar sliding over the bar A, the difference of the velocities of any two touching points in B and A is a velocity parallel to the slide-surface, or 'guide-surface.' Thus, the velocity of the bar A being known, the velocity of any point in B can be obtained by adding to the velocity of *any* touching point in A a velocity parallel to the guide-surface, and further adding a velocity perpendicular to the line joining this touching point with the point in B whose velocity is to be found.

Four-bars with sliding-joint

This last added component is that due to the *rotation* of B in the field of the base-plate. If the touching surface of B has the same shape as that of A, so that B always 'fits' on to all parts of A into contact with which it comes, and if during the sliding these fitting surfaces are forced always to lie close together, then the angular velocity of B is different from that

of A only in consequence of the sliding that occurs in the slot. In this case, if the velocity of A be completely known, the linear velocity of any point in B can be calculated by adding to the velocity which the point would have if B were rigidly attached to A, a velocity parallel to the *nearest part* of the guide-surface—that is, perpendicular to a normal let fall from the point on the guide-surface.

In the illustration (Fig. 71*a*) the velocity of point A round P_1 is supposed to be known, and it is plotted as $p\,a$. The slot is a straight one. Then $p\,\beta$ and $a\,\beta$ are drawn perpendicular to $P_1\,B$ and A B. This gives $p\,\beta$ the velocity that point B would have if the cross-head were rigidly attached to the guide-bars, and if $\beta\,b$ be drawn parallel to the slot the point b must lie in this last line. But B is guided by the radius rod $P_2\,B$. Therefore, $p\,b$ is drawn perpendicular to $P_2\,B$ to meet βb in b; then $p\,b$ is the velocity of B in the P field, and $\beta\,b$ is the velocity of sliding in the slot. In Fig. 71*b* the slot is curved. There is first found $p\,\beta$ in same manner as in Fig. 71*a* from the known velocity of A, and $p\,b$ is drawn perpendicular to $P_2\,B$. From β is drawn $\beta\,b$ parallel to the guide-surface at C, the foot of the perpendicular from B on the guide surface. Note that C is an actual point of the guide-surface only if the curvature of this latter be uniform. If it be not uniform, then C must be taken in the extension of the circular arc at D which is coincident with the small portion of the guide-surface over which actual contact takes place. Such guide-surfaces in mechanisms are commonly of uniform curvature, because this is an evident condition of continuous close fit between the two sliding bars. **Four-bars with sliding joint**

If a block C (see Fig. 72) slide in two slotted bars A and B, the first of which has a translatory velocity $p\,a$, and the second a translatory velocity $p\,b$, evidently the method of finding the velocity $p\,c$ of the block is to draw from a and b two lines parallel to the two slots in A and B. If these lines meet in c, then $p\,c$ is the velocity required.

If the slotted bars have rotational instead of purely translatory velocities, then precisely the same construction is to be followed, making $p\,a$ and $p\,b$ the linear velocities of the touching points of the guide-surfaces in A and B. Now, however, it is evident that one and the same block cannot constantly fit close to both slotted guide-surfaces. But if two fitting blocks, one fitting the one slot and the other the other, be pinned together, then the above construction may be applied to find the velocity of the centre of the joining pin, and from the velocity of this centre it is easy to deduce by methods already explained the velocities of all other points in the two sliding blocks.

Four-bars with sliding-joint

18. The following application (see Fig. 73) of the construction for sliding motion to toothed-wheel gear well illustrates the complete generality of the method.

The sketch represents four wheels, $P_A A$, $P_B B_1$, $P_B B_2$, and $P_C C$, pinned to the base-plate at P_A, P_B, and P_C. The point A of the first touches the point B_1 in the second, the two surfaces having here a common tangent to which the line $(A\,B_1)\,T_{AB}$ is drawn normal. The third wheel being mounted on the same shaft as the second, these two are to be looked upon as forming, along with the shaft, one bar of the mechanism. The third and fourth wheels touch at the common point $(B_2\,C)$, and the line $(B_2\,C)\,T_{BC}$ is drawn normal to the common touching surface. The points T_{AB} and T_{BC} are in the centre lines $P_A P_B$ and $P_B P_C$.

Toothed gear

The velocity of the wheel A, and therefore of its touching point A, is supposed known, and this velocity is marked off as $p\,a$ from any pole p, the line $p\,a$ being drawn perpendicular to $P_A A$. Then $p\,b_1$ and $a\,b_1$ intersecting in b_1 are drawn perpendicular to $P_B B_1$ and to $B_1 T_{AB}$. This gives $p\,b_1$ the velocity of B_1 and $a\,b_1$ the velocity of sliding of one tooth over the other.

Then $p\,b_2$ and $b_1\,b_2$ intersecting in b_2 are drawn perpendicular to $P_B B_2$ and to $B_1 B_2$; $p\,b_2$ is the velocity of the point B_2. Finally, $p\,c$ and $b_2\,c$ intersecting in c are drawn perpen-

dicular to $P_C C$ and to the normal $C T_{BC}$. This gives $p\,c$ the velocity of C, and $b_2\,c$ the sliding velocity of this second pair of teeth over each other. The process may be carried on indefinitely through a whole train of wheel work, however complicated. As a method of finding the velocities throughout such a train, however, it is not a practically useful one, because the directions of the normals to the touching surfaces cannot be very accurately obtained on the drawing unless the 'pitch points' T_{AB}, T_{BC}, &c., are known, and if these are known to start with the various velocities can most simply be determined from them directly without reference to the touching points.

The point T_{AB} may be looked on as indicating two points—one in the first wheel, which may be called T_A, and the other in the second, which will be called T_B. To obtain the velocity of T_A, the triangle $p\,a\,t_a$ is constructed similar to the triangle $P_A\,A\,T_A$. In this triangle $a\,t_a$ coincides with the line $a\,b_1$, and $p\,t_a$ is perpendicular to $P_A\,P_B$. Making a similar construction for the velocity of the point T_B, we find that the point t_b coincides with the point t_a. Thus the points T_A and T_B in the two wheels have the same velocity $p\,t_{ab}$, and the point T_{AB} is therefore called the 'pitch point.' The angular velocities of the two wheels are therefore inversely as the distances $P_A\,T_A$ and $P_B\,T_B$, this being a familiar theorem proved in the ordinary treatment of toothed gearing. Similarly, if $p\,t_{bc}$ be drawn perpendicular to the centre line $P_B\,P_C$ to its intersection with $b_2\,c$, this $p\,t_{bc}$ is the velocity of the pitch point T_{BC} of the pair of wheels (B C). If the teeth be so shaped as to give constant angular velocity ratios between the wheels, and the angular velocity of any one wheel of the train be kept constant, the points T_{AB}, T_{BC}, &c., in the diagram of the mechanism and the points t_{ab}, t_{bc}, &c., in the velocity diagram remain fixed throughout the periodic motion of the train. It may also be noticed that since

Toothed gear

$$\frac{a\,t_a}{p\,t_a} = \frac{\mathrm{A\,T_A}}{\mathrm{P_A\,T_A}} \text{ and } \frac{b_1\,t_b}{p\,t_b} = \frac{\mathrm{B_1\,T}}{\mathrm{P_B\,T_B}},$$

therefore

$$\frac{a\,t_a}{p\,t_a} \cdot \frac{p\,t_b}{b_1\,t_b} = \frac{a\,t_{ab}}{b_1\,t_{ab}} = \frac{\mathrm{A\,T_A}}{\mathrm{P_A\,T_A}} \cdot \frac{\mathrm{P_B\,T_B}}{\mathrm{B_1\,T_B}} = \frac{\mathrm{P_B\,T_B}}{\mathrm{P_A\,T_A}},$$

so that $\frac{a\,t_{ab}}{b_1\,t_{ab}}$ also measures the ratio of the angular velocity of wheel A to that of wheel B. The condition that the angular velocity ratio should remain constant may thus also be expressed by the condition that the line $a\,b$ in the velocity diagram, representing the velocity of sliding of tooth over tooth, should be divided in a constant ratio by the fixed point t_{ab}. [This point t_{ab} is only fixed if the angular velocities themselves, as well as their ratio, remain constant, the magnitudes of these angular velocities being proportional to $p\,t_{ab}$.] Whether this proposition can be utilised in simplifying the practical drawing out of the teeth-profiles, so as to secure a constant velocity ratio, the author has not yet investigated.

Toothed gear

This kinematic method has been applied to many actual mechanisms. The diagrams when fully drawn are to a certain degree complicated, although not difficult to produce; and in order that a large number of points should be distinctly marked and lettered on each of the various curves, the diagrams should be drawn to a moderately large scale. This fact prevents any of these diagrams being usefully reproduced in the atlas to this book; but the student should not fail to work out for himself a sufficient number of practical examples to familiarise him with the method.

CHAPTER X.

STATIC STRUCTURES, FRAMES, OR LINKAGES.

1. We have already described the construction of the 'Moment Diagram' for parallel forces, and in Chapter VIII., Fig. 57, this construction is extended to non-parallel co-planar forces. The construction consists in drawing a connected series of lines across the spaces lying between the force-lines, the joint-points of the lines lying on the joint-lines of the spaces, and the lines being drawn parallel to the radii drawn from a 'pole' to the joints of the vector-addition diagram. This series of lines across these spaces forms a closed polygon if the group of forces 'balance' both with regard to vector sum and with regard to moment. This state of 'balance' involves two distinct conditions: firstly, that the vector sum is zero, the graphic test of which is that the vector-addition diagram forms a closed polygon, and the physical meaning of which is that this group of forces when applied to any portion of material has no influence in changing the velocity of the centre of inertia of this material; secondly, that the sum of the moments of the forces is zero round any and every axis, the graphic test of which is that the moment-diagram forms a closed polygon, and the physical meaning of which is that the application of this group of forces to any mass does not result in any change of the angular velocity of this mass round any axis whatever. Ill-taught students are apt to think that when the vector sum is zero the moment must necessarily also be zero round every axis, because a resultant of zero magnitude can have, it is supposed, no moment.

Conditions of balance

M 2

In doing so, they forget the case of force-couples, which have a moment although the vector resultant has zero magnitude.

Conditions of balance

The locor resultant in this case is not zero; it is two equal and opposite forces. If the vector resultant is zero, and the moment diagram does not close, then the locor resultant is a couple. This is indicated by the first and last lines of the moment diagram being parallel.

2. Confusion also sometimes arises from the fact that a group of forces which balances and has no moment round any axis still has a 'moment-diagram' from which one measures bending moments produced by this group of forces. It is to be remembered that these bending moments are due to partial sets only of the whole balancing group. Thus the height of

Moment-diagram

the diagram at any point measures the moment round an axis lying in the line on which the height is measured, of all those forces of the group that lie on one side, either right or left, of that line.

When the forces are not parallel, it was explained that this polygon was inconvenient when used directly as a 'moment-diagram,' although it always remains of the greatest utility in finding the moments because it supplies the resultants of the various partial groups of forces.

3. We must now look upon this closed polygon in another light, namely, as a skeleton drawing of a jointed linkage which

Closed chain

would keep its shape and position—i.e. be generally in equilibrium, under the action of the forces. Looked on in this light, it is sometimes called a 'chain,' sometimes a 'linkwork,' or a 'linkage.' The latter term is preferred by the author.

4. A stiff 'link' may for our purpose be defined as an approximately rigid piece of material forming one part of a

Definition of link

structure or machine and jointed to the other parts in such a way that the forces exerted between it and those other parts act in lines passing *through definite and easily ascertainable points in the link*, which points are called the 'joints' of the

link.[1] The word 'point' must not here be understood in its mathematical sense. No forces act through mathematical points; they are all more or less distributed through certain volumes or masses and act always through sectional areas of finite magnitude. When one speaks of a force acting through a point, one either refers to the point through which passes the centre-line, or line of the resultant (as previously defined), of a group of distributed forces; or else one means that the distributed force is concentrated through a very small mass and acts on a very small sectional area.

Definition of link

5. If two links be connected by a cylindrical pin-and-eye joint in which the pin fits the eye loosely, each link acts on the pin over a very small portion of its circumference; and if there be very little friction between the pin-and-eye surfaces, then the force, being nearly exactly normal to the surface, passes very nearly through the centre of the pin. Whether the pin be loose or tight, provided there be very little friction, the pressure must be almost symmetrically distributed on either side of a line passing through the pin-centre, and, therefore, the centre-line of the pressure passes approximately through the pin-centre.

Flexible joint

Thus a rigid mass connected to other pieces of material only by pin-and-eye joints is a 'link' according to the above definition, because it is definitely known that it is acted on only by forces whose lines pass through the centres of the pins; pass through those centres, at any rate, with a degree of approximation quite equal to that of the other data assumed in the course of engineering calculations. For example, our knowledge of the loads on the structure or machine is always far from accurate, as is also our knowledge of the strength and the modulus of elasticity of the materials out of which the structure or machine is built.

[1] We will need afterwards to admit into our investigations flexible, extensible, and compressible links; but in the meantime, and except when specially mentioned as flexible, &c., a 'link' will here be understood ot mean an 'approximately rigid' link.

6. As applied to actual structures, the accuracy of this idea is interfered with in the first place by the friction between pin and eye. In the case of machine-joints the coefficient of friction is always small and may usually be known with considerable exactitude. In this case it is not difficult to take into account the effect of friction in the graphic calculations of the forces; the method of doing so will be explained in a subsequent chapter devoted to the dynamics of machines. In the pin-joints of roofs, bridges, and similar static structures, the coefficient of friction may often be very large, and the friction, moreover, may be increased very much beyond the product of the *effective force* through the joint and the coefficient of friction in consequence of the pin being driven in tight. Also, since there is no relative motion between the links, it is impossible to say in which direction the friction at each joint is active. By laborious calculation of the stretching and contraction of the various links under stress, it would be possible to discover in which direction the friction at each joint acts during the process of erection of the structure, that is, when the stresses are first produced. But the friction during the permanent life of the structure is not necessarily the same as that during erection, and it may evidently be reversed at various joints many times in consequence of to-and-fro straining due to rise and fall of temperature and variation of load. Thus it becomes practically impossible to take account of joint friction in these structures; it may be theoretically possible to do so, but, as a matter of fact, it is never done, and the labour of the operation would be out of proportion to the utility of the result. It is useful to remember, nevertheless, that, however great the friction may be, it can never shift the centre-line of the force through the joint away from the centre of the pin so far as the radius of the pin, except in the case of a pin driven in extremely tight. Probably in actual structures the deviation seldom exceeds two-thirds the radius of the pin.

Stiff joints

7. Secondly, it must be noticed that the forces applied to

certain members of these structures are, instead of being concentrated, distributed over nearly their whole length. If the pieces be so large that their own weights must be taken into account, then these weight-forces are distributed throughout the whole mass of each piece. Again, the weight of a roadway borne by the booms of the girders is usually, and should always be, applied to these booms as a well-distributed load. Similarly, the rafters of a roof carry the weight of the roof covering as a distributed load.

In constructing the force-diagrams for such structures, these distributed loads are usually supposed replaced by concentrated loads applied at the joints, each whole load being divided into parts in proportion to the distances of the joints from the centre-line or resultant of the load. This substitution, although it is usually employed in the applications of graphic statics, is by no means necessary, and in what follows there will be shown the method of constructing the diagram without using this artifice. The disadvantage of using this method of substitution is that, in following it, one loses sight of the bending moments produced by the distributed loads. The method hereafter explained will give in the diagram these bending moments as well as the direct forces along the joint-lines. **Distributed load**

8. Thirdly, it may be here remarked that the graphic method applicable to pin-joint structures is very commonly applied to structures built up of members riveted together, there being many rivets in each joint. Even if there were only one rivet at each joint, the method would not be strictly applicable, because such a joint is *stiff*—that is, is capable of resisting relative rotation of the jointed links, or, in other words, capable of transmitting a bending moment. **Moments at joints**

9. If there are many rivets in the joint, the magnitude of the bending moment the joint is capable of transmitting is proportionately great, and the application of the method becomes proportionately objectionable. A link pin-joint may **Stiffness of joint**

be distinguished from other joints (otherwise than as in the above definition) by saying that it is incapable of transmitting a force-couple. Any set of co-planar forces may be reduced to a force and a couple, and by varying the magnitude of the couple the position of the force-locor can be changed at will; because the resultant of a force and a couple is an equal and parallel force shifted in position a distance dependent on the magnitude of the couple. The sum of the forces across any section of a bar may thus be reduced to a force through the geometrical centre of the section and a force-couple. To produce a force-couple there must be opposite stress-forces exerted at different parts of the section, and the magnitude of the couple will be proportionate to the distance apart of those parts of the section where the opposite stresses are exerted. Thus, if the cross-dimensions of the section or of the joint be small, it is incapable of transmitting a large bending moment,

Stiffness of joint

and the resultant force through it can have its line passing only at a proportionately small distance from the geometrical centre of the section or joint. Otherwise expressed, the section or joint has an amount of *flexibility* or *rotational freedom* which diminishes as its cross-dimensions increase. The greater the flexibility or rotational freedom of the joints of a structure, the greater is the accuracy of the force calculations made by the graphic method of this chapter on the assumption of approximately frictionless pin-joints. The degree of error at any joint involved in the adoption of this method is measured by the magnitude of the force-couple the joint is capable of exerting divided by the vector-force-sum actually transmitted. This quotient is the maximum possible deviation of the line of this force-resultant from the centre of the joint, through which centre the resultant is assumed to act. Some examples of the possible errors involved in applying the method to riveted-joint structures will be given subsequently. Evidently the error occurring at strut joints is apt to be greater than that at tie-bar joints, because struts are necessarily made stiff

to avoid buckling, while flexibility is rather a merit than a disadvantage in a tie-bar.

10. When a pin-joint link has only two joints in it, then the link is acted on by two forces only. If the link be kept in balance under these two forces, they must be equal in magnitude and opposite in direction in order that they should give zero vector-sum, and they must act along one and the same line in order that they should not form a force-couple which would give the link an increasing angular velocity. Since each of them acts through one of the joint-centres, this common line of action must be that joining the two joints in the link. This line is called the 'joint-line' or the 'centre-line' of the link. The statics of structures built up of two-joint links only is a great deal simpler than that of structures in which occur links with more than two joints. These simpler structures will, therefore, be exclusively studied in the first place. Subsequently the method will be extended to more complicated cases.

Two-joint links

11. In Fig. 57, Chapter VIII., looking on the series of lines (P) A B C D E as the centre lines of the links of a closed chain of two-joint links, the links P A and P B will push the pin at the joint (P A B) in the directions *p a* and *b p*, and if these latter represent the magnitudes as well as the directions of the pushes of these links, then the pin will be kept in balance by the three forces *p a*, *a b*, and *b p*, since these three form a triangle. Both these links are in compression. Now if *b p* be the magnitude of the push of the link B P on this pin, this same link will push the pin at its other joint (P B C) in the *opposite* direction *p b* with an equal force, provided the link is subject to no acceleration of velocity. This latter pin will then be in balance if it also receive a thrust from the link C P in the direction and of the magnitude *c p*, the three balancing forces being *p b*, *b c*, and *c p*. Similarly, the pin (P C D) will then be in balance if *d p* measure the pull exerted on it by link D P, this link being in tension and pulling the next pin (P D E)

Single-pen Linkage

in the direction $p\,d$ opposite to that of its pull on pin (P C D). $e\,p$ must now represent the thrust of the compression link E P in order that the pin (P D E) should be in equilibrium, and on this supposition this same link will exert the opposite force $p\,e$ on the pin (P E A) at the point I. But the link A P has been assumed to be in compression to the amount $a\,p$, and thus the three forces $a\,p$, $p\,e$, and $e\,a$ will keep this last pin balanced. The pins being originally motionless, and being kept motionless by the balance of these sets of forces, the whole linkage is also kept in balance—i.e. keeps its shape and position unchanged, because the positions of the joints define that of the linkage as a whole.

Single-pen Linkage

12. So far the reasoning shows that the shape of the linkage is consistent with the supposition that it may be kept in equilibrium under the given group of forces. It will so rest in equilibrium if the above stress-forces arise in the links. It remains to be examined how and why these stresses do actually arise when the external forces are applied. Suppose the linkage to be given this shape before these external forces are applied, the links being all unstressed. Since there are no stresses in the links at the instant of application of the forces, it is evident that at this first instant the above balance cannot exist, and the first effect of the application of the external forces at the joints is that the pins begin to move through at least minute distances in the directions of the applied forces. Since these forces form a balancing system, and, therefore, do not move the centre of mass of the linkage, the small motions of the joints are in different directions and produce changes of length of the joint-lines of the links. The material of the links is resistant, and these changes of length, or strains, cause stresses to arise in the links of such character as to resist the further motion of the joints. The further the joints move, the greater necessarily become the strains in the links and the resisting stresses increase in magnitude towards the above limits which produce balance at the joints, and which

Origin of stresses

limits are actually reached provided they are not beyond the ultimate breaking strengths of the links. The extent of the movements of the joints and of the strains of the links that actually occur during the application of the load before this equilibriated condition is reached—that is, the general deformation or strain of the linkage as a whole that is necessary before balance can be attained, depends on the extensibility and compressibility of the various links. These latter depend on the modulus of elasticity of the material and upon the sections of the links. In linkages containing three-joint links it depends also on the flexibility of the links. The greater the extensibility, compressibility, and flexibility of the links, the greater is the general strain needed before balance is attained, and balance is never attained if the links do not possess sufficient strength to exert the stress-forces shown by the diagram.

Origin of stresses

In the above it has been attempted only to explain the general way in which the stresses arise. It has not been attempted to show that the small motions of the joints due to the application of the external forces are exactly such as will produce the precise strains needed to develop the stress-forces that will produce balance. It has not been even attempted to show that these stresses as they develop bear to each other the correct proportions as shown on the balancing stress-diagram whose pole is p. In such an investigation one would need to take into account the distribution of mass throughout the linkage as well as the distribution of sectional stiffness; because during this adjustment to the condition of equilibrium one has to do with unbalanced forces producing acceleration of momentum. This difficult and complicated investigation is unnecessary, firstly, because our graphic calculations will be limited to such frames the links of which are sectionally so stiff as to prevent the actually occurring strains being large enough to produce any sensible alteration in size or shape of the structure; and, secondly, because in actual engineering

structures the loads are applied almost invariably very gradually, so that at no instant of the erection and loading does there exist at any part any considerable unbalanced force.

Origin of stresses

Again, in all practical cases in which the linkwork is in itself flexible—i.e. capable of taking different forms under the action of different sets of forces, as in the case of a simple chain such as that of Fig. 57—we do not find it set out in the shape suitable for equilibrium before the loads are applied. It has at the outset some other shape, and, when loaded, not being in balance, it is drawn into a new, possibly an entirely different, shape such as may (or may not) produce balance.

13. Now there are two cases to be distinguished carefully from each other. The distinction depends on the shape possessed by the linkage in relation to the particular forces applied to it. If the shape be such that, when the forces are applied, the linkage is drawn by the action of the forces further and further away from the shape which would harmonise with balance under these forces, then the loading will result in the complete collapse of the linkwork. If, on the other hand, the forces draw the linkwork nearer and nearer the configuration in which it would balance under these forces, then, conversely, stability will be the result. A similar distinction is to be made between two kinds of *equilibriating* shapes. If the balancing forces be slightly altered or disturbed in any way and the linkage have in consequence its shape slightly changed, and if then the external forces be exactly restored to their former magnitudes and directions; then, if the shape be of the one kind, the linkage will come back to its original position and shape, but will deviate still further from these if it be of the other kind. The first kind of shape is said to be one of 'stable' equilibrium, and the second of 'unstable' equilibrium. Thus, if the links be arranged in the form of an arch and the loads be directed in the general direction more or less towards the centre of the arch, then collapse will take place under the disturbing con-

Stability

ditions described. But if the form be that of a suspension bridge or inverted arch, and the loads act away from the centre of curvature, then after the disturbing influence has passed away the linkage will tend to draw back to its original shape, and is, therefore, in stable equilibrium. **Stability**

14. As there can be drawn an infinite number of moment or locor-summation diagrams for the same set of balancing forces, so there is an infinite variety of shapes and of sizes of linkages which would keep in balance under the same set of forces. The joints of all these possible linkages lie, of course, on the same force-lines. The possible variety of shape may be expressed as before by saying that the pole of the force-diagram may have any position, and the size may be varied without shifting this pole. If the order be settled in which the spaces between the force-lines are to be crossed by the links, then the possible variety of linkage may be otherwise expressed by saying that three consecutive joints may be chosen anywhere on their respective force-lines; or that the direction and position of one link and the direction of one contiguous link may be arbitrarily selected. There are evidently *three* elements of arbitrary choice: as soon as these are chosen the external forces determine the rest of the lines of the structure, and at the same time the stress-forces in all the links are fixed. **Three points variable**

15. Let the full lines of Fig. 74 represent a set of balancing forces acting on such a balanced linkage and the corresponding force-diagram with its pole p corresponding with the inside space or pen P. At the joint (P 7 1) or E we have the external force $_{7\,1}$ balanced by the forces $_1p + p_7$ exerted by the links 1 P and P 7, the first of which is in tension and the second in compression. **Six points variable**

On the two links P 7 and P 1 choose any two positions B and A, and suppose the linkage altered by the abolition of the remaining lengths, B E and A E, of these links and the insertion of three new links A B, B D, and D A, jointed at B and A

to the links P 7 and P 1, and also jointed at any point D on the force-line 7 1. We will now prove that the linkage so altered will still be in equilibrium under the same set of forces.

From 7 and p in the force-diagram draw $7p'$ and pp' parallel to B D and A B. Then at the joint B we have the force $p\,7$ exerted on the pin by the link P 7. This link may be kept in balance under this force and the forces exerted by links B D and A B. These latter are in the directions of these lines, and, therefore, must equal $7p'$ and $p'p$ in order to preserve the balance. The pin at D is now acted on by the forces $p'\,7$ (exerted by link D B and opposite to the thrust of this link at its other end B) and 7 1. It can only be kept in balance by a force $\#\,1p'$, and this force can only be exerted by the link D A. It has to be proved, therefore, that $1p' \parallel$ D A. If this be proved, then the pin at joint A will also be in balance, because it will be acted on by the three forces $1p$ (already determined as the stress in the link 1 P of the original unchanged linkage) and pp' and $p'\,1$, which three form a triangle in the force-diagram. The intersection of $1p$ and $p'\,7$ we will call q, and that of P 1 and B D we will call C.

Six points variable

The triangle $pp'q$ has its sides parallel to those of the triangle A B C; therefore,

$$\frac{pq}{qp'} = \frac{\mathrm{AC}}{\mathrm{CB}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The triangle $q\,1\,7$ has its sides parallel to those of the triangle C E D; therefore,

$$\frac{1q}{q7} = \frac{\mathrm{EC}}{\mathrm{CD}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

The triangle $pq7$ has its sides parallel to those of the triangle E C B; therefore,

$$\frac{pq}{q7} = \frac{\mathrm{EC}}{\mathrm{CB}} \text{ or } \frac{pq}{q7} \times \frac{\mathrm{CB}}{\mathrm{EC}} = 1 \quad . \quad . \quad . \quad . \quad (c)$$

Therefore,

$$\frac{\mathrm{AC}}{\mathrm{CD}} = \frac{pq}{qp'} \times \frac{\mathrm{CB}}{\mathrm{EC}} \times \frac{1q}{q7} \text{ from } (a) \text{ and } (b)$$

$$= \frac{1q}{qp'} \times \frac{pq}{q7} \times \frac{\mathrm{CB}}{\mathrm{EC}} = \frac{1q}{qp'} \text{ from } (c).$$

But since A C $\parallel$ 1 q and C D $\parallel$ $q p'$, therefore angle A C D = 1 $q p'$.

The triangles A C D and 1 $q p'$, having these angles equal and the sides containing them in the same proportion, are, therefore, similar; and these sides being parallel, the remaining side A D in the one triangle is parallel to the corresponding side 1 p' in the other. This is what had to be proved.

For the simple linkage of one pen P only we have now substituted a linkage of two pens. The second pen may be called P′; and the links A B, B D, and D A may now be called P′ P, P′ 7, and P′ 1.

Six points variable

The first pen is now (P) 1 2 3 4 5 6 7 P′, and corresponding to it we have in the force-diagram the pole p from which radiates the pencil (p) 1 2 3 4 5 6 7 p', the radii of which are parallel to the sides of the above pen. The second pen is (P′) 1 P 7, and corresponding to it there is the pencil (p') 1 p 7, whose radii from the pole p' are parallel to the sides of the pen.

In commencing the construction of the original one-pen linkage, or say that of the pen P of the new two-pen linkage, we saw that there were three elements left to the arbitrary choice of the designer—that is, rather, left to be chosen from considerations unconnected with the problem of obtaining equilibrium in the linkage under the given external forces. In constructing the second pen P′ we find that three more elements of arbitrary choice have been supplied. Thus all three points A B D may be chosen arbitrarily anywhere on the three lines P 7, 7 1, and 1 P; or the direction and position of one of the three new links and the direction of a second may be so chosen.

A linkage of one pen for a given set of loads thus leaves the designer with a free choice of three elements. A linkage divided into two pens leaves him free to choose six elements.

It can easily be shown that the link dividing the whole into two pens can be introduced at any place. Thus in Fig. 74 it might stretch from a joint such as A in P 1 to a joint at any position in link P 5. With the link A B as taken above, the forces through links P 7 and P 1 added together as locors equal the locor resultant of all the external forces applied to the linkage to the one or the other side of the dividing link A B. These two forces taken each in one direction —namely, in the directions in which they act on the joints A and B—equal the locor resultant of the forces 1 2, 2 3, 3 4, 4 5, 5 6, and 6 7; taken in the opposite direction-- that is, as they act on the joints (P 6 7) and (P 1 2)—they equal the single force 7 1, which is the only force to the right of the dividing link A B. But in investigating the balance of the pins at the new joints A and B we had only to consider the substitution of the forces through the links A B, B D, and D A, for the stress-forces along B E and A E; and it is evidently of no consequence to the proof whether these last two added as vectors equal a single actual force 7 1 or the resultant of a number of forces. It is thus clear that the dividing link can be placed in any position in the linkage; and indeed this can be proved directly following for any special case exactly the model of the proof given above, the number of steps, however, and the complication of the proof increasing directly in proportion to the number of forces acting on, or to the number of joints in, the new pen.

Six points variable

16. We can again operate similarly on the pen P of the two-pen linkage, splitting it once more by another dividing link, say from a joint in P 2 to a joint in P 3, or in fact stretching between any two points in the sides of the pen. The whole linkage will then become a three-pen linkage, and, for a given set of loads, there will be nine elements in the

Variability and complexity

construction of this three-pen linkage at the arbitrary choice of the designer. There will be a third pole in the force-diagram, from which will radiate force-lines parallel to the links bounding this third pen.

17. The introduction of each new dividing link, which means the splitting up the whole linkage into one additional pen, gives, as already said, three new elements of arbitrary choice in the construction of the linkage. But it must be noted that if the new dividing link be inserted between two already existing joints, its insertion in that particular position amounts to the choice of two of these three elements—namely, the positions of the joints of the new link—and there remains only one element to be chosen capable of giving a new outline to the linkage as a whole. Thus, if A and B were placed at the joints (P 1 2) and (P 6 7), the change of form effected by the creation of the new pen resolves itself into the single choice of the position of D on the line 7 1.

Limitation

18. A most important point to notice is that the change of shape of the linkage made on one side of the newly introduced link does not necessitate (i.e. in order still to maintain the condition of equilibrium under the given set of forces) any alteration of shape on the other side of this link. It is important to note this, because it means a *first step towards stiffness* in the linkwork. Either portion of the linkage to the right or to the left of the dividing link—that is, either of the pens P and P′—can be changed in shape at will in a multitude of various ways. Any change in any part of one pen affects the shape of the whole or most of that pen, but does not affect that of the other pen; except, of course, in the case of the change being effected by displacing the dividing link, which is common to both. Thus, the shape of each pen is independent of that of the other. The shape of the linkage at any joint is defined by the angles between the links meeting at that joint. *Complete stiffness* is attained when the shape at the joint between each pair of jointed links is independent of

Degree of stiffness

the shape at every other part. So long as the shape at one part depends on that at other parts (the loading forces remaining the same), the linkage is to a certain degree *flexible.* The flexibility is diminished and the stiffness increased the more the whole is split up into numerous pens. At the same time, a larger number of elements of free choice remain open—that is, the designer of the structure has a freer hand in shaping it as he thinks fit. At a certain limit this multiplication of the pens produces *complete stiffness*, and at the same limit the shape of the structure comes completely within the power of the constructor to make it what he likes. That is, the general shape and outline of the structure will no longer depend in any degree on the external forces applied to it; if these forces form a balancing group of forces, then the structure will be in equilibrium whatever shape be given to it; while, if these forces do not form a balancing system, then, of course, the structure cannot be brought into equilibrium under their action by giving it any particular shape. This limit will be explained presently. The effect of introducing more dividing links than are needed to produce this limit of subdivision is to cause what is called 'redundancy,' a term the meaning of which will be afterwards fully expounded.

Degree of stiffness

19. The relations between the linkage and the force-diagram are extremely interesting from a purely geometrical point of view. It was Clerk Maxwell who first investigated them in the fullest and most general manner. For instance, the point p' may be chosen anywhere, as may also one of the three points A B C, say A on its line P 1. Then the lines A B, A D, and B D being drawn parallel to the three radii from p' to p, 1 and 7, the locus of the intersection of the two latter lines is the line 7 1 in the linkage diagram. On account of a certain reciprocity in the geometrical relations between the two diagrams, Clerk Maxwell called them 'Reciprocal Figures.' But a purely mathematical view of any matter is the last thing an engineering student should allow himself to take.

Relations of locor- to vector-diagram

Fig. 75 will serve to illustrate the properties most important in connection with the use of these diagrams for the investigation and design of engineering structures.

20. Each of the pens A B C D E F has a corresponding pole in the force-diagram, which poles are given the similar names *a b c d e f*. From each pole radiates the same number of force-lines as there are links bounding the corresponding pen. These lines are parallel to these bounding links, and measure to scale the stress-forces exerted by the links. Each link separating two pens is common to both pens, and the corresponding line in the force-diagram, therefore, joins the poles of these two pens. The separating link may be called the joint between the two pens. The joint between the two poles, therefore, measures the force exerted by the joint between the pens. Throughout this book the pens will be named by capital letters, A B C, &c., and the corresponding poles by similar small letters. For instance, from the pole *b* radiates a pencil of 5 lines (*b*) *a*, 3, 4, *c*, 10, parallel to the 5 links bounding the pen (B) A, 3, 4, C, 10 and measuring the forces through these links. **Nomenclature**

The spaces outside the linkage and separating the lines of external load will be named by large figures, 1 2 3, &c., and the corresponding points in the force-diagram by similar small figures. These spaces are not pens proper, because they are not completely surrounded or closed by definite boundaries; towards one side they are open and of indefinite extent. But the force-diagram contains a point corresponding to each of these external spaces, from which point radiate the same number of lines as the space has definite bounding lines, and these radii are parallel to these bounding lines and measure the forces exerted along them. Therefore, these points may consistently be termed the poles corresponding to the external spaces, and the external spaces may be called 'outside pens' or 'open pens' or 'unclosed pens.' **Outside pens**

21. The outside pens are separated by the locor lines of

the external loads. The distinction between these joints (or dividing lines) of the outside pens and those of the internal pens is not very simple to explain, but ought to be thoroughly grasped. It is still more important to understand the essential similarity between these two kinds of joints, and this can be best done by recognising clearly their one point of difference.

Each stress line corresponding to a link of the structure is evidently used as a force line twice over in considering the balance of the joint-pins. At one end of the link it represents a force acting on the pin in a certain direction; in considering the balance of the pin at the other end it represents an equal but opposite force. If it be in compression, the link uses the pin at its one end as its abutment for the thrust it exerts on the pin at its other end. If it be in tension, each pin acts as the anchorage of the link for the pull exerted on the other pin. But the line of the stress-diagram representing the action of an external force, and corresponding to the joint between two external pens, is used only once and in one direction only. It is so, at any rate, so far as the stress-diagram is used for investigating the forces throughout the structure. This then is the distinction: the one line is a *stress* line representing two opposite forces; the other is a *force* line, to be used only in one direction, since it represents only one force.

Outside and inside links

But this external load is simply one aspect of a stress action between the structure under investigation and its material surroundings. Suppose it to be the supporting force exerted by a pier on a bridge. Then this is only one aspect of the compression stress in the pier. The pier acts as a strut between the bridge and the earth; it is a link connecting the bridge with the rest of the material world outside the bridge. At one end this link has its abutment on the earth; at the other end it abuts against the bridge. It is, therefore, to be properly called an *external or outside link*—i.e. outside the structure under investigation. As the line of pressure through

a pier does not always, or generally, coincide with the geometrical axis of the pier, it may frequently be preferable to call it an 'outside link line' rather than 'outside link,' meaning thereby that it is the force-centre-line, or the centre-line of the force through the outside link; but when no ambiguity arises, the shorter phrase may be legitimately used. The links belonging to the structure itself will be called, in contradistinction to these, 'inside' or 'internal links,' this designation *including* the peripheral or boundary links which separate the inside from the outside pens.

Outside and inside links

When the 'external force' is due simply to the weight of a mass resting on or hanging from the structure, there is no material link, such as a pier, communicating the stress between the structure and the outside material world. The stress bond in this case is the attractive force of gravity between the loading mass and the earth. But whether we consider this attractive force to be exerted across absolute empty space, or to be a real active stress existent in that portion of the substance of the radiant-energy-carrying ether lying between the attracting masses,[1] we may take this space intervening between the earth and the heavy load as representing an outside link along which the external stress is transmitted, the line in which the weight acts being the centre-line of the external stress. When this weight-load is applied to the under side of a structure, this imaginary or representative external link wil lie wholly outside the structure, and will be a tie-bar. When the load is applied to the upper boundary of the structure, the direct external weight-link to the earth would pass downwards through the structure, thereby confusing the spaces into which the link-lines divide the whole diagram. It will

[1] This latter is the belief of nearly all modern physicists. To those who believe the doctrine of conservation of energy it would appear impossible to account for the conversion of active into so-called 'potential' (better called 'latent') energy, except by supposing the latter to exist in some form of stress-energy filling the space intervening between or surrounding the attracting bodies.

be found later that it is important to have all the external link-lines lying clear of, and outside the structure. In this case, therefore, it is best to adopt an artificial convention, and imagine the weight to be due to a compressive stress through an outside link (a strut) lying above the structure. This outside strut must be supposed to have its abutment at its upper end against an arch built up from the earth and over the structure.

Fig. 76 will help to explain these conventions.

Outside and inside links

The outside links 11, 1, and 6, 7, are compression links consisting of the end piers or abutments of the bridge. 7, 8; 8, 9; 9, 10; and 10, 11 are tension links due to loads resting on the bottom boom. 1, 2; 2, 3; 3, 4; 4, 5; 5, 6 are external compression links due to loads on the top chord.

No essential distinction, therefore, exists between internal links and external links. The forces of all the external links taken as acting upon the structure must form a balancing group of forces if the structure is to remain *in situ*. These forces, taken as acting on the earth—i.e. on the world external to the structure—also form a balancing set of forces.

Method of sections

22. Take any section $\alpha\beta$ through the structure. We may now consider the portion to the left of this section as a structure by itself, held in balance by a set of stresses between it and the external world. The outside links are now 9, 10; 10, 11; 11, 1; 1, 2; 2, 3; 3, F; F, G; and G, 9. The latter three are internal links of the whole bridge, but are now to be considered as external links of the structure consisting of that part of the bridge to the left of the section line $\alpha\beta$. This shows still more clearly how non-essential is the distinction between internal and external links.

This latter set of external stresses, including 3 F, F G, and G 9, must form a balancing system. From this fact the stresses 3 F, F G, and G 9 can be deduced from a knowledge of the rest of the group 9, 10; 10, 11; 11, 1; 1, 2; 2, 3. The two conditions of vector and locor balance are sufficient to determine the unknown magnitudes of those three forces, whose directions

are known. This method of finding the stresses in members of structures is called the 'Method of Sections,' and will hereafter be shown to be of great utility. The graphic construction by which it is effected has already been explained in Section 14 of Chapter VIII.

Method of sections

23. If the pens of a structure be all triangular, the structure is evidently 'stiff,' in the sense that its shape cannot be altered except by lengthening or shortening the individual links. The members of engineering structures either actually having, or being intended to have, such sectional sizes as prevent the tensive and compressive forces coming upon them from altering their lengths in any degree enough to change appreciably the general shape of the structure, the above is the sense in which the term 'stiff' is technically applied to such structures.

Stiffness

24. In order to effect complete stiffness it is not necessary that all the pens should be triangular. In building up a plane structure, in order to locate 'stiffly' each new joint as it is added to the design or in the actual erection, it is evidently necessary and sufficient to add *two* new two-joint links.

Thus, in adding any number of joints, double the same number of links must be added.

Criterion in plane frames

Since in the simplest possible structure—viz. a triangular one-pen truss—there are three joints and three links, that is, there are three less links than double the number of joints, therefore the general formula connecting the number of joints with the number of links in a completely stiff but non-redundant plane linkage is

$$l = 2j - 3$$

where j = number of joints and l = number of links.[1]

[1] This condition is sometimes expressed as a relation between the number of pens and that of joints—namely, the number of pens is two less than the number of joints. But in some structures the pens overlap each other and the counting of their number becomes confusing. Again, in 'solid' as distinguished from plane structures, the corresponding rule connecting the numbers of links and of joints is of easy application, whereas that connecting the numbers of joints and of enclosed volumetric pens is confusing and difficult to apply.

Thus in Fig. 76 the number of joints is 11, and $2 \times 11 - 3 = 22 - 3 = 19 =$ number of links. In Fig. 78 the number of joints is 41, and $2 \times 41 - 3 = 82 - 3 = 79 =$ number of links. In Fig. 80 there are 18 joints, and $2 \times 18 - 3 = 36 - 3 = 33$ links.

This only applies to flat structures built of two-joint links. If the linkage contain 'beam-links'—i.e. links capable of resisting cross bending and containing more than two joints—then evidently in order to locate stiffly in a beam-link a third, fourth, fifth, &c., joint, no further links need be added to the structure. Thus the above formula can be applied unaltered to structures containing beam-links, provided that in counting j only the end joints (or any *two* joints) of each beam link be included in the count. In the diagrams in this book beam-links will be distinguished from others by being indicated by double lines; one line being thin and representing the centre-line of the link, and the other being thick.[1] Thus in Fig. 77, counting only two joints to the horizontal beam, the number of joints is 7, and $2 \times 7 - 3 = 11$ is the number of links, counting the beam as one link only. This structure is, therefore, stiff in the sense of the word we are now using. In the ordinary meaning of the word, of course, its degree of stiffness will depend on the rigidity of the beam.

Criterion in plane frames

25. In a 'solid' structure—i.e. one of three dimensions as distinguished from a flat one—in order to locate each new joint stiffly it is necessary to add 3 new links. The simplest possible solid linkage is a pyramid with a triangular base. This has 4 joints and 6 links—i.e. its number of links is 6 less than 3 times the number of its joints. Therefore, the general formula for a stiff and non-redundant solid linkage is

Criterion in solid frames

$$l = 3j - 6$$

where j is the number of joints and l the number of links.

[1] This system is consistent with the representation of ties and struts by thin and thick lines respectively, because one side of a beam is in tension—i.e. acting as a tie—while the other is in compression.

As in the previous case, in counting j only two of the joints of each beam-link are to be included in the count, and the beam is reckoned as one link only.

26. If the number of links be greater than that given by the above formulas, the linkage is 'redundant,' and it becomes impossible to determine all the stresses in it either by algebraic or graphic means unless account be taken of the elastic extensions, contractions, and bending of the ties, struts, and beams. **Redundancy**

27. If the structure rest on the earth by means of fixed connections, so that the joints between structure and earth are approximately immovable, then evidently the earth forms one link that goes towards the stiffening of the structure; and a number of links in the structure itself (exclusive of the earth link) one less than given by the above rules is sufficient for stiffness. Thus, an arch hinged at centre and at abutments becomes a stiff structure only when considered in connection with the earth, which performs the function of a tie-bar whose tension balances the horizontal thrust through the arch. If a structure stiff in itself be connected with the earth by fixed joints, by this connection it is converted into a redundant structure, the stresses in whose members can only be determined by taking account of the elastic deformation of the structure and of the earth abutments. If, however, the joint with the earth have a certain degree of freedom; if, for example, a stiff non-redundant girder rest at one end on horizontal roller-plates that leave its one end free to move horizontally without any retarding horizontal force worth taking account of; then the directions of the supporting forces at the earth joints become determinate, and the problem of finding the stresses throughout the structure can be always completely solved. **Earth connections**

28. To ensure stiffness and non-redundancy the structure taken as a whole must fulfil the conditions expressed by these rules; but their fulfilment for the whole structure is not an **Closer analysis**

infallible criterion of stiffness and non-redundancy, nor is it necessary that a stiff and non-redundant structure should fulfil these conditions in each of its separate parts. Thus, a structure may be flexible in one part and redundant in another, in such a way as to leave it flexible as a whole and yet fulfil as a whole the rules. For example, a quadrilateral with braces across both diagonals is redundant, and to one side of this may be joined another quadrilateral with braces across neither diagonal. The second quadrilateral is flexible and the whole linkage is flexible, but it fulfils the conditions, there being 6 joints and $2 \times 6 - 3 = 9$ links.

To test completely, therefore, the stiffness of a proposed design of a complicated character some additional criterion is necessary. This consists in taking a number of sections, each dividing the structure in two parts. The parts must then fulfil certain conditions wherever the section be taken if the structure be stiff and non-redundant. These conditions are the following:

Closer analysis

If the section pass through a joint and *one* bar; or if without passing through a joint it cuts through *three* bars; then in each of the two parts the usual rule ($l = 2j - 3$) must be fulfilled, the joint cut through in the former case being included in the count for *each* part.

The section cannot pass through two bars only if the linkage be stiff, unless the two bars are actually jointed together.

If the section pass through a joint and two bars, or through no joint and through four bars; then in one of the two parts the usual rule ($l = 2j - 3$) must be fulfilled, while in the other part there must be one degree of flexibility (i.e. $l = 2j - 4$). This rule is exemplified in Figs. 78, 80, and 86. If the section be through a joint, the joint has to be included in the count for each part.

In no case are the bars cut through by the section to be included in the count of links for either part.

These conditions must be fulfilled for every possible section.

In these tests, in counting the number of links cut through by a section a beam-link is to count as two. Thus, a section of a stiff structure cannot pass through two links only if both be two-joint links, but it may do so if one of them be a beam-link, as may be seen from Figs. 77, 87, 88, 89, and 90. **Closer analysis**

29. The method of finding the stresses by constructing at each joint a graphic diagram showing the balance of the forces acting on the pin at that joint has been in use for a long time. But this method is clumsy; it necessitates the drawing of each line twice over, because the stress in each link acts on two pins, and, therefore, enters into two of the above diagrams. The modern improved method of stress diagrams consists in combining all these into one diagram, so that no line, or as few as possible, need be drawn twice over.

It is not always possible to avoid repeating the same stress in the diagram, but in a good diagram this repetition is minimised, and in almost the majority of cases it does not need to occur at all. Its avoidance depends altogether upon *taking the stress lines in the correct order* in building them into the diagram. Referring to Fig. 75, take for example the stress in the the link 5 C. This stress gives one of the forces acting on the joint-pin (5 C B 4). It, therefore, appears in the closed vector polygon for this joint; and thus it must appear in connection with the stress 4 5, i.e. in the same polygon. But it must also appear in similar connection with 5 6, because it forms one of the balancing forces on the joint-pin (5 6 D C). It acts in opposite directions on these two pins, and, therefore, in the one diagram it will be read 5 *c*, and in the other *c* 5, and it must radiate from the joint of stress lines 4 5 and 5 6, because these latter have to be in similar connection with each other since they form part of the polygon showing the vector balance of the group of external forces. Similarly, 6 *d* must radiate from the joint of 5 6 and **Cyclic order**

6 7. Therefore, in the diagram for joint (C 5 6 D) the stresses along C 5, 5 6, 6 D, must follow each other either in this order or the exact reverse. Since the three follow in this manner and there are only four altogether in the polygon for this joint, it results that the same must hold for the four. That is, that in the stress-diagram the four must follow in the same order as the locor lines are cut in the linkage in circulating round the joint. The circuit of the joint may be made either right-handedly or left-handedly—that is, either as (C 5 6 D) or as (C D 6 5). Such order is called 'cyclic' order, and there are two possible cyclic orders that may be followed in building up the stress-diagram, the right-handed or the left-handed. *But the same cyclic order must be preserved throughout the stress-diagram for the whole structure.*

Cyclic order

30. The cyclic order that has been adopted in building up the stress diagram must be marked on it either in words under the title, thus, 'Left-handed' or 'Right-handed,' or very much better graphically by a circled arrow-head thus ↺ or ↻. By so marking it, it becomes quite unnecessary to place any arrow-heads on any of the lines of the diagram (neither on the external nor on the internal force lines) to indicate force directions. In the link diagram to distinguish compressions from tensions it becomes unnecessary to use either arrow-heads or + and —. Thus, in Fig. 75, which is marked ↻, in order to find whether the link D E is in tension or compression, it is only necessary to notice that on the lower joint (9 C D E F) the stress on this link will be read *d e*, and on referring to the stress-diagram we find that this is *downwards, or towards the joint*—that is, the link is thrusting downwards on this pin, and is, therefore, in *compression*. Similarly, link C D is in tension, because at the lower joint its force is to be read *c d*, and this we find to be directed upwards, or *away from the joint*.

Tension or compression?

31. Arrow-heads placed on the lines in the linkage are very objectionable because they are ambiguous. Thus, a link

marked —<————>— would by some people be read as being in tension, the arrows being taken to indicate that the ends of the link tend to separate; while others would understand that the arrows indicate the directions of the forces exerted by the bar on the pins at its opposite ends, and therefore read the bar as being in compression.

Arrow-heads

32. The signs + and − affixed to the link-lines are not ambiguous, and are in fact very commonly and usefully employed. When the stresses are read off in figures, these figures should be written along the links, and a plus may be prefixed for tensions and a minus for compressions.

Use of + and −

33. But a readier and more quickly perceived distinction between tension and compression members may be made in inking in the linkage by employing two colours; for example, by inking all the tie-bar lines in black and the strut lines in red. In this book all tie-bars will be shown by thin black lines, and strut lines by thick black lines. The stress-diagram may very suitably be inked in blue. In this book, however, fine black lines are used for the stress-diagrams.

Two colours. Thick and thin lines

34. The cyclic order of the stress-diagram applies to the vector summation diagram of the outside forces. The same order as holds for each individual joint is that in which these external forces must be arranged to follow each other in the stress-diagram. The order here refers to that in which the outside links are cut through in making the circuit outside the whole structure, either right-handedly or left-handedly. Thus, in Fig. 75, they will be found in the stress-diagram in the order 1 2 3 4 5 6 7 8 9 10, the same as that in which the outside pens are traversed in making a right-handed circuit outside the linkage.

Constancy of cyclic order

The same cyclic order will be found to be preserved in the vector-summation diagram showing the balance of the forces acting on any individual pen. Thus, the pen (C) 9 B 5 D in Fig. 75, consisting of 4 links, is kept in balance under the action of the links 9, 10; 10 B; B 4; 4 5; 5 6; 6 D; D E;

E F; and F 9. This is the order in which these links are cut in making the right-handed circuit of the pen, and in the stress-diagram it will be found that the corresponding forces follow each other in this order and form a closed vector polygon. Here C is the 'pen-joint' of the above links, and we see that the same cyclic order holds for the balance of 'pen-joints' and for that of 'point-joints.'

Constancy of cyclic order

Taking two pens together, for example C and D of Fig. 75, and considering them as a compound or complex pen, we find the same law holds; thus, these two are held in balance by the forces in the stress-diagram 9, 10 *b* 4 5 6 7 *e f* 9, these being arranged according to the right-handed cyclic order in which the corresponding spaces are traversed in making the external circuit of these two pens. The same will be found to hold true of any combination of pens one can take. Thus, the application to the group of external forces under which the whole structure is in balance is only a special application of a general law. The same law again holds good for the balance of *individual links*. Thus, the link C D is in balance under the forces shown in the stress-diagram by 9 *c* 5 6 *d e f* 9, this order showing the correct direction of these forces.

Thus pins, links, simple pens and complex pens are all subject to this one law of the same cyclic order in the stress-diagrams exhibiting their balance. The importance and utility of this law in the art of constructing graphic stress-diagrams cannot be overrated.

Loads at internal joints

35. There is an important exception to the rule that in the stress-diagram no stress-line is to be repeated twice. In the examples hitherto given the external forces are assumed to act at external joints, and there will be found at once a difficulty in following the method explained if they do not do so. But the loads are actually sometimes applied at internal joints. For example, Fig. 78 shows a 'stiff' arched linkage in which the load, which is the weight of a roadway, rests on the bars B M, D O, F Q, &c., &c., and is supposed to be con-

centrated at the joints along this line. In order to get over the difficulty arising in this way, the best and, so far as the author knows, the only correct method is to suppose added a number of extra links lying along the lines of the external forces and terminating at joints in the outer periphery of the linkage. In Fig. 78 these imaginary links are shown by dotted lines running from the joints where the loads are applied to the upper boundary or chord. They might as simply have been drawn downwards to the lower chord, in which case the stress-diagram would have taken a slightly different shape from that shown but would have given exactly the same stress-magnitudes. It is generally a matter of theoretic indifference towards which boundary the imaginary links are inserted, but it is of practical importance to place them so that they shall intersect as few actual links as possible. As in Fig. 78, it is generally unnecessary to allow each to intersect more than one actual link—namely, that at the outer boundary; but in Fig. 86 there are examples of them crossing two actual links. At this intersection a new joint must be imagined to be inserted, which joint will divide the actual boundary link into two. The supposition of the insertion of one imaginary link will thus increase by two the total number of links in the linkage and by one the total number of joints.[1] Thus, the question of the flexibility, stiffness, or redundancy of the structure will not be affected by the introduction of these imaginary links. This is still the case whether or not the imaginary links intersect more than the one peripheral actual link. At each 'internal' intersection of imaginary and actual links there must be supposed introduced a new joint. Each such joint will split the imaginary and the actual links each into two parts, thus increasing the number of links by two and the number of joints by one.

Loads at internal joints

Thus in Fig. 78 the load actually applied at the joint (O P Q F E D) is supposed transferred to the upper chord by

[1] The whole of this argument refers only to *plane* linkages.

the imaginary link E E′. This divides the space E into two spaces E and E′. The link joining joint (C D E 11) with (E F G 10) is divided by the imaginary joint (E E′ 10, 11) into two links, E′ 10 and 11 E. The supposed insertion of the imaginary link leaves the stresses through these latter two links entirely unaffected—that is, it leaves them exactly the same as if the imaginary link and joint were not inserted; because at this joint we have jointed together four links lying in two straight lines. The vector diagram showing the balance of this joint-pin must, therefore, be a parallelogram, and the stresses through the two links lying along the same line are, therefore, equal and quite independent of the magnitude of the stress in the other pair of links lying along the second line. In the present case the parallelogram is a rectangle and is seen on the stress-diagram to be *e e′* 10, 11. Thus the stress condition of the link (11, 10) E is unaffected by the supposition of the insertion of the extra link. Furthermore, the stress on the imaginary link E E′ being necessarily equal to the external load, the thrust of this link on the joint-pin (O P Q F E D) is equal to that actually exerted by the load; and the supposition made, therefore, leaves the stress condition at this joint entirely unaffected.

Loads at internal joints

This artifice is, therefore, legitimate, since it alters no actual stress conditions, and its adoption eliminates all difficulty in constructing the stress-diagram according to the ordinary simple procedure already explained.

36. The result, however, is that a number of lines in the stress-diagram appear twice over. Thus all the 'external stresses' applied at internal joints are repeated; each appearing once in the external-force vector-balance polygon and once as an internal stress, namely, that through the imaginary link. Every actual link that is divided into two by imaginary joints has also the line representing the stress through it repeated twice in the diagram. This repetition is unavoidable. Although it increases the number of lines and thus

Stress-lines repeated

decreases the 'readability' of the diagram, it can never lead to any ambiguity or real difficulty in reading off results.

37. When two actual links cross each other without being actually jointed at their intersection, the same artifice must be adopted in order to make it possible to draw out the stress-diagram without confusion and without break of continuity. An imaginary joint must be supposed inserted at their intersection. This will divide each of the links into two. The stress-diagram for the joint will be a parallelogram, and the stress on each of the two links will be repeated twice. The spaces separated by the links should be lettered so as to give similar names to the two stress-lines representing one and the same link stress. Thus, in Fig. 79, we have a lattice girder of common design except that vertical struts across the quadrilateral spaces are usually introduced. In this design the diagonals that run from top to bottom chord are not actually jointed at their crossings, but at these points joints must be imagined for the purpose of enabling one to draw out the stress-diagram in an orderly and continuous fashion. **Crossing links**

38. If the spaces are lettered after a system similar to that shown in Fig. 79, the identical stresses in the stress-diagram will appear with names, not the same, but with a distinctive similarity to each other. Thus, C_3 C_4 and D_3 D_4 are really parts of one and the same link, but they are treated as if they were two separate links. Their names have, however, this in common, that the suffices 3 4 are identical in both. These suffices in the name of each differ from each other— that is, 3 differs from 4; while the letters appearing in each do not differ, as for instance in C_3 C_4 only the one letter C appears. The stresses of these two links will appear as c_3 c_4 and d_3 d_4; and these will be opposite sides of a parallelogram, and are identical in name so far as the suffices 3 4 are concerned. The other pair of equal sides of this same parallelogram will be c_3 d_3 and c_4 d_4, whose names are identical so far as the letters go, although the suffices differ in the two stresses **Lettering**

while in each stress only one suffix number appears. These are the stresses in the links $C_3\ D_3$ and $C_4\ D_4$, whose names have the same sort of identity, and which are imagined to be separate links, although in reality they are one and the same. Using this system of lettering, no two bars which are really different bars will be found to have names that are not distinctive. In each name either the two letters are the same while the suffices differ, in which case the suffices are to be taken as the part of the name individualising the real link or the corresponding stress; or else the two suffices are the same number while the letters differ, in which case the letters are to be taken as the distinctive part of the name.

Lettering

The linkage in Fig. 79 has 7 joints in each chord, or 14 in all. In order to be stiff it requires, therefore, $2 \times 14 - 3 = 25$ links. It has actually 26, and is, therefore, redundant in one degree, and its stress-diagram cannot be drawn without taking account of its elasticity.

39. Fig. 80 shows a non-redundant stiff linkage of similar design. There are here at the ends 4 joints at each of which only two links meet. At each such joint no stresses can exist in the links unless an external force acts at the joint.

Thus at the joint $(9\ J_2)$ no outside link acts, and therefore the stresses in the two internal links, $J_2\ 9$ and $9\ J_2$, are zero. It must be understood that this rule applies only to 2-joint links. If one of the pair of links be a beam, then force may be transmitted through the joint.

2-link joints without load

If, again, the two links meeting at such a joint lie in the same direction, then it follows, not that their stresses are zero, but that they are equal, both being subjected to equal pulls or to equal thrusts.

40. Another case, frequently of use in surmounting special difficulties in complicated structures, is that of a joint at which three links meet, two of which links lie along the same line. Here the force-component exerted normally to the links in line by the link not in line must be equal and opposite to

3-link joints, 2 in line

the component of external force acting on the joint resolved in the same direction. This enables one to find the stress on the oblique link at once without working up to this joint from other parts of the structure; in other words, it may be possible to begin drawing the stress-diagram from this joint.

In many roofs as actually put up such 3-link joints occur on the lower chord where no load is applied. The joints A B C D in Fig. 81 are examples of this. The stay-bars at these joints cannot be stressed except by the weight of the tie-bars of the lower chord, and are, therefore, practically of no use whatever. If the lower chord were composed of two stiff beams, then the stays would be of use. **3-link joints, 2 in line**

41. In the case of an external force acting on such a 3-link joint, if we look on this force as that of an external link, the joint is a 4-link one. Thus, at *any* 4-link joint two of whose links lie in one line, the force-components of the other two perpendicular to the first two must be equal and opposite. It not infrequently helps one out of a special difficulty in proceeding with the stress-diagram to remember this obvious fact.

Again, if a known external force acts at such a 4-link joint, the component of external force perpendicular to the two links in alignment must balance the algebraic sum of the similar components of the forces along the two other links. Thus in Fig. 80 all except the end joints are of this character. Therefore, the algebraic sum of the vertical components of the forces of any pair of diagonal braces meeting at a joint must equal the vertical component of the load at that joint. The student should inspect the stress-diagram to trace out the truth of this proposition for each of the joints of this structure. **4-link joints, 2 in line**

42. In drawing the stress-diagram of a plane structure, the general rule is that one must begin at a 2-link joint—i.e. one of two *internal* links. The outside link makes a third, and, *its stress being known*, the triangle of balancing forces can at once be drawn. From this one must proceed to a *con-* **Order of procedure**

tiguous 3-link joint; a 4-link joint would leave three unknown forces which it would be impossible to determine. The third joint taken in building up the diagram may be a 4-link one, but must not be 5-linked. At each successive step the new joint attacked must be one at which *all the forces except two* are either known as data or have already been found by the stress-diagram construction. It is here that the beginner generally goes wrong; he goes prematurely to a joint where more than two forces remain undetermined, and, being unable to solve this, becomes confused and perhaps disheartened. It is only in special circumstances, however, which are dealt with below, that he has not at each step a choice of one or more joints fulfilling the condition necessary for solubility.

Order of procedure

Nearly all structures have at least two joints at each of which meet only two inside links. It is a matter of theoretic indifference at which of these two the stress-diagram is begun. Sometimes one will be preferable as a matter of convenience to the draughtsman. Usually, the commencement having been made at one of these two joints, the other is the *last* to be dealt with at the completion of the diagram, and the last two joints usually furnish the graphic test of accuracy of draughtsmanship. Sometimes, however, it is convenient to work part of the diagram from one of the 2-link joints, and another part from the other similar joint, the two parts being eventually joined somewhere near the centre of the diagram, and this junction then affording the test of accuracy.

Number of links at a joint

43. The normal distribution of links at the various joints is the following: two 2-link joints, two 3-link joints, all the others 4-linked. This may be recognised from the formula $(2n - 3)$ links for the n joints of a stiff non-redundant plane frame. Each link goes to two joints, and, therefore, there are $(4n - 6)$ link-ends at the n joints. The above distribution accounts for $\{2 \times 2 + 2 \times 3 + (n - 4)4\} = 4 + 6 + 4n - 16 = 4n - 6$, the same as just mentioned.

The two 2-link joints can never be found contiguous to

each other in a stiff linkage. The two 3-link joints are generally contiguous to the two 2-link ones.

If there be one 5-link joint, then there are three 3-link ones. If there be several 5-link joints, there is a corresponding extra number of 3- or 2-link ones. Each 6-link joint leads either to an extra 2-link instead of a 4-link joint, or to two extra 3-link instead of 4-link joints. Each extra degree of complexity in any one part of the structure beyond the normal arrangement in which the most complex joint has four links leads to a corresponding amount of extra simplicity in another part. **Number of links at a joint**

44. There may also be no 2-link joints along with six 3-link ones, all the rest having four; in this case the count of the link-ends being

$$6 \times 3 + (n - 6)\,4 = 18 + 4n - 24 = 4n - 6$$

as before. Figs. 85 and 86 give examples of this distribution. It is here impossible to start drawing the stress-diagram in the usual way. One method of surmounting the difficulty is to start by help of the 'Method of Sections.' Another is to use the 'Method of Two Trials and Two Errors.' Explanations and examples of both methods are given below. **No 2-link joints**

45. These principles of stress-diagram construction will now be illustrated by going through in detail a few examples.

First, take the arched linkage of Fig. 78. The structure is symmetrically built, but the load is not symmetrically distributed.

The loads are $2, 3 = 12, 13 = 2$ tons each;

$$3, 4 = 4, 5 = 5, 6 = 6, 7 = 7, 8 = 8, 9 = 4 \text{ tons each};$$
$$\text{and } 9, 10 = 10, 11 = 11, 12 = 7 \text{ tons each};$$

Stiffened Arch

all these loads being vertical and parallel to the axis of symmetry of the linkage. The supporting force $13, 1$ is inclined at $45°$ to the horizontal. These are the load data.

A left-handed stress-diagram is adopted: the outside pens are numbered in left-hand cyclic order; the loads applied at internal joints are transferred to peripheral joints by means of imaginary links; and the inside pens are lettered as shown.

Next, the loads are plotted to scale consecutively from 2, 3 at the right-hand end to 12, 13 at the left-hand along a vertical straight line lying near the left-hand end of the paper. The scale actually adopted is $\frac{1}{8}'' = 1$ ton; but this is made small for the sake of engraving. In practice $\frac{1}{5}'' = 1$ ton would be better. From 13 in stress-diagram a line is drawn at 45° to horizontal. Any pole is chosen, and parallel to the radii from it are drawn the lines of a simple link polygon through the spaces 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 of the linkage.[1] The lines through spaces 2 and 13 are produced to meet, and through their intersection a vertical line is drawn, this being the centre-line of the loads. The line 13, 1 of the supporting force is drawn through the joint (13, 1 K) at 45° to meet this centre-line. The line to this last intersection from joint (1, 2 K) gives the direction of the other supporting force; and this force is now drawn from 2 in the stress-diagram to meet 13, 1, thus determining the point 1.

Stiffened Arch

The following table now shows the order in which the stress-diagram is built up. In the force-diagram for each joint certain force lines (all except two) have been drawn already before the completion or closure of the diagram. The last two, or those requiring to be drawn when the diagram is completed for the joint, are indicated in the table by a bar being placed over them:—

Name of joint in linkage	Balanced force-diagram	Name of joint in linkage	Balanced force-diagram
13, 1, K	$13, 1, \overline{k, 13}$	T 1 U	$t\,1\,\overline{u\,t}$
K 1 M	$k\,\overline{1\,m\,k}$	12, 13, K M B	$12, 13, k\,\overline{m\,b\,12}$
M 1 N	$m\,\overline{1\,n\,m}$	12 B C	$12\,\overline{b\,c}\,12$
N 1 O	$n\,1\,\overline{o\,n}$	11, 12 C C′	$11, 12, c\,\overline{c'\,11}$
O 1 P	$o\,1\,\overline{p\,o}$	C′ C B M N O D	$c'\,c\,b\,m\,n\,o\,\overline{d\,c'}$
P 1 Q	$p\,1\,\overline{q\,p}$	11 C′ D E	$11\,c'\,\overline{d\,e}\,11$
Q 1 R	$q\,1\,\overline{r\,q}$	10, 11 E E′	$10, 11, e\,\overline{e'\,10}$
R 1 S	$r\,1\,\overline{s\,r}$	E′ E D O P Q F	$e'\,e\,d\,o\,p\,q\,\overline{f\,e'}$
S 1 T	$s\,1\,\overline{t\,s}$	10 E′ F G	$10\,e'\,\overline{f\,g}\,10$

[1] The simple linkage to determine the supporting force 12 is omitted from the engraving for clearness' sake, but was, of course, drawn in the original diagram.

Name of joint in linkage	Balanced force-diagram	Name of joint in linkage	Balanced force-diagram
9, 10 G G′	9, 10 *g g′* 9	1 **S T**	1 **s t** 1
G′G F Q R S H	*g′ g f q r s h g′*	1 **T U**	1 **t u** 1
9 G′H I	9 *g′ h i* 9	V′V J U 1 **U J**	*v′ v j u* 1 **u j v′**
8 9 I I′	8 9 *i i′* 8	7 V′ **J I**	7 *v′* **j i** 7
I′I H S T U J	*i′ i h s t u j i′*	6 7 **I I′**	6 7 **i i′** 6
8 I′J V	8 *i′ j v* 8	**I′ I J U T S H**	**i′ i j u t s h i′**
7 8 V V′	7 8 *v v′* 7	6 **I′ H G**	6 **i′ h g** 6
1 2 **K**	1 2 **k** 1	5 6 **G G′**	5 6 **g g′** 5
1 **K M**	1 **k m** 1	**G′ G H S R Q F**	**g′ g h s r q f g′**
1 **M N**	1 **m n** 1	5 **G′ F E**	5 **g′ f e** 5
1 **N O**	1 **n o** 1	4 5 **E E′**	4 5 **e e′** 4
1 **O P**	1 **o p** 1	**E′ E F Q P O D**	**e′ e f q p o d e′**
1 **P Q**	1 **p q** 1	4 **E′ D C**	4 **e′ d c** 4
1 **Q R**	1 **q r** 1	3 4 **C C′**	3 4 **c c′** 3
1 **R S**	1 **r s** 1	**C′ C D O N M B**	**c′ c d o n m b c′**

This completes the drawing of the diagram with the exception of the line **b** 3. But both the points **b** and 3 are already located, and the line can therefore be drawn by joining them without reference to the direction of the line **B** 3 in the linkage. The test of the accuracy of the drawing consists in finding whether these two lines in linkage and stress-diagram be parallel as they ought to be.

Stiffened Arch

Another form of test (although not an independent test) is to locate point **b** once more by drawing the force-diagram for the joint 3 **C′ B**. If the drawing be accurate, the two **b**'s thus found will coincide. A third mode of locating it is to draw the diagram for the joint **M K** 2 3 **B**. If the inaccuracy be found to be much, it shows that some downright mistake has occurred in forming the diagrams; if it be small, it shows that no such mistake has been committed, but that strict parallelism between the lines has not been maintained, or that the lines have not been drawn strictly through the right points.

There being no external loads at the joints along the upper chord where the diagonals meet, and the diagonals having equal, although contrary, inclinations, the stresses in

the two members of each pair of diagonals meeting at an upper joint are equal in magnitude and of contrary sign. Also the stresses in the pair of horizontal members meeting at each upper joint differ by the arithmetical sum of the horizontal components of the stresses in the two diagonals—i.e. differ by double the horizontal component of one of these diagonal stresses.

Stiffened Arch

46. The next example taken will be Fig. 80. Here we need to use a number of imaginary joints on account of the crossing of the bracing-bars. The structure is symmetrical, and is supported by abutments at the four points Q P S T. At the latter point T the supporting force is known to be zero, and, since no load acts here, the stresses on the links J_2 9 and 9 J_2 are zero. The loads 1 2, 2 3, 3 4, 4 5, 5 6, 6 7, 7 8, 8 9 are equal and inclined 30° to the horizontal. The loads 10, 11; 11, 12; 12, 13; 13, 14; 14, 15; 15, 16; 16, 17; 17, 18; 18, 19 are equal and vertical. The whole load on the upper chord 1, 9 is in magnitude half the whole load on the lower chord 10, 19. The supporting force 20, 1 is supposed known in direction as at 45° to the horizontal, and also in magnitude as $\frac{3}{4}$ that of 1, 9. The supporting force 19, 20 is known to be inclined at 60° to the horizontal. These are the data.

Compound Warren girder

The right-hand cyclic order has been adopted in constructing this diagram. The stress-diagram is begun by plotting to a suitable scale 20, 1 and 1 2 3 4 5 6 7 8 9; then vertically downwards from 9 to 19′ the loads on the lower chord. The location of these latter is only temporary, as the unknown force 9, 10 has been omitted from its proper cyclic position, and has to be inserted subsequently after being found. A pole is chosen, and parallel to radii from it are drawn the links of a simple one-pen linkage through the outside pens 20, 1 2 3 4 5 6 7 8 (9, 10) 11, 12, 13, 14, 15, 16, 17, 18, 19 : 9 and 10 being taken as one space because of the omission of the force 9, 10; and the latter links being drawn

parallel to the radii from the pole to the points 11′ 12′ 13′. . . . 19′. This simple polygon is omitted from the engraving. The first and last link through spaces 20 and 19 are produced to meet, and through their joint is drawn R the line of the resultant ‖ 20, 19′. The known supporting force line 19, 20 through P is produced to meet R, and from their joint is drawn a line to Q. This latter line gives the direction of the supporting force 9, 10, and parallel to it is drawn from 19′ the line 19′, 19 in the stress-diagram. In same diagram from 20 is drawn 20, 19 ‖ 20, 19 in linkage—i.e. at 60° to horizontal. This gives the final position of 19 in the stress-diagram. The parallelogram 19, 19′, 9, 10 is completed by drawing 9, 10 and 19, 10. The line 9, 10 is now properly located in the stress-diagram, giving the magnitude of the supporting force at Q. The lines 9, 19′ and 19′ 19 may now be rubbed out for clearness' sake, and the lower chord loads are plotted off along the line 10, 19.

Compound Warren girder

47. It is frequently the case, as here, that the joint of the load resultant line with one supporting force line lies far outside the paper. The difficulty of drawing from this joint to another point such as Q is got over by a construction such as that shown in the figure. The point r is the joint of R and line P Q. From any other point r'' in R directions to P and Q are taken, and lines $r\,p'$ and $r\,q'$ are drawn parallel to them. p' is the joint of $r\,p'$ and 19, 20. From p' is drawn $p'\,q'$ ‖ P Q, and its joint q' with $r\,q'$ is a point in the desired line 9, 10 through Q. The proof is obvious. Owing to similarity of triangles,

Joints beyond the drawing

$$\frac{\mathrm{P}\,r}{p'\,r'} = \frac{r\,r''}{r'\,r} = \frac{r\,\mathrm{Q}}{r'\,q'},$$

and therefore the three lines $\mathrm{P}\,p'$, $r\,r'$ and $\mathrm{Q}\,q'$ converge to the same point. The difficulty may be otherwise avoided by the construction explained in Section 13 of Chapter VIII.

48. In the following table, describing the further construction of the stress-diagram, a shorter method of writing out

the process is adopted. In each new joint-diagram we here leave unnamed the lines already drawn, and mention only the two new lines to be drawn. The joint is also left unnamed. The new point or pole determined by the joint of the two new stress-lines is placed last in the triangular name given to the process of finding it. Thus $\genfrac{}{}{0pt}{}{3}{b_2}\!> b_3$ means that two lines are to be drawn from the already known points 3 and b_2, and that their intersection gives the point b_3. It is, of course, to

Contracted description of construction

be understood that these two lines are to be drawn parallel to the lines 3 B_3 and $B_2 B_3$ in the linkage. By help of this system the whole process for Fig. 80 is accurately expressed in the following table. The consecutive order in which the table is to be read is along the horizontal lines from left to right as in ordinary reading.

$$\genfrac{}{}{0pt}{}{18}{20}\!> b_1 \quad \genfrac{}{}{0pt}{}{20}{2}\!> a_2 \quad \genfrac{}{}{0pt}{}{b_1}{a_2}\!> b_2 \quad \genfrac{}{}{0pt}{}{17}{b_2}\!> c_2$$

$$\genfrac{}{}{0pt}{}{b_2}{3}\!> b_3 \quad \genfrac{}{}{0pt}{}{c_2}{b_3}\!> c_3 \quad \genfrac{}{}{0pt}{}{16}{c_3}\!> d_3 \quad \genfrac{}{}{0pt}{}{c_3}{4}\!> c_4 \quad \genfrac{}{}{0pt}{}{d_3}{c_4}\!> d_4 \quad \genfrac{}{}{0pt}{}{15}{d_4}\!> e_4$$

$$\genfrac{}{}{0pt}{}{d_4}{5}\!> d_5 \quad \genfrac{}{}{0pt}{}{e_4}{d_5}\!> e_5 \quad \genfrac{}{}{0pt}{}{9}{11}\!> i_1 \text{ and } \genfrac{}{}{0pt}{}{i_2}{j_2} \left\{\begin{matrix}\text{coincident}\\ \text{with}\end{matrix}\right\} \genfrac{}{}{0pt}{}{i_1}{\text{and } 9}$$

$$\genfrac{}{}{0pt}{}{8}{i_2}\!> i_3 \quad \genfrac{}{}{0pt}{}{i_2}{12}\!> h_2 \quad \genfrac{}{}{0pt}{}{i_3}{h_2}\!> h_3 \quad \genfrac{}{}{0pt}{}{7}{h_3}\!> h_4 \quad \genfrac{}{}{0pt}{}{h_3}{13}\!> g_3 \quad \genfrac{}{}{0pt}{}{h_4}{g_3}\!> g_4$$

$$\genfrac{}{}{0pt}{}{6}{g_4}\!> g_5 \quad \genfrac{}{}{0pt}{}{g_4}{14}\!> f_4 \quad \genfrac{}{}{0pt}{}{g_5}{f_4}\!> f_5$$

The points e_5 and f_5 are now determined. The line joining them should be $\parallel E_5 F_5$, that is, vertical. This is the test of the accuracy of the drawing.

Symmetrical Roof

49. When the structure is symmetrical and the distribution of load also symmetrical, the two axes of symmetry coinciding, the stress-diagram is also symmetrical, its axis of symmetry being perpendicular to that of the linkage and of the loading. This leads to much simplification in the construction of the diagram.

This is illustrated by Fig. 82, which shows a simple roof

truss with vertical loading and vertical supporting forces. From the symmetry of the load distribution it is evident that the two supporting forces are equal, and, therefore, it is unnecessary to draw in a simple linkage in order to determine them. Each is equal to half the whole downward load. The lines 5 6, G **G**, and 14, 15 lie on the axis of symmetry. Right-hand cyclic order is used.

The loads are 1, 2 = 9, 10 = 2 tons; 2, 3 = 8, 9 = 3 tons; 3, 4 = 7, 8 = 4 tons; 4, 5 = 5, 6 = 6, 7 = 6 tons;

11, 12 = 12, 13 = 13, 14 = 14, 15 = 15, 16 = 16, 17 = 17, 18 = 3 tons.

The whole downward load is, therefore, 57 tons, and each supporting force is $28\frac{1}{2}$ tons. These outside stresses are plotted to a suitable scale along the vertical line 1, 10, 11, 18, 1. **Symmetrical Roof**
The horizontal dotted line at the middle of this vertical line is the axis of symmetry of the stress-diagram. The construction proceeds in the following order:—

$$ {}^{18}_{2}{>}a \quad {}^{a}_{3}{>}b \quad {}^{17}_{b}{>}c \quad {}^{c}_{4}{>}d \quad {}^{16}_{d}{>}e \quad {}^{e}_{5}{>}f \quad {}^{15}_{f}{>}g \quad {}^{14}_{g}{>}\mathbf{g}. $$

There is no need of completing the other half, as the results would only be a repetition of those already obtained. The only test of accuracy is that the line *g* **g** should be bisected by the axis of symmetry. If the diagram were completed each pair of points with similar letter names (i.e. poles of similar right- and left-hand pens) should be equidistant from the same axis and should be vertically one over the other. Each pair would thus give a test of accuracy similar to that given by *g* **g**.

50. In Fig. 83 we have again a symmetrical structure, namely, a railway bridge with roadway along the bottom boom. **Symmetrical Bridge. Distributed rolling load**
The first stress-diagram is for a symmetrical load—viz. a uniformly distributed load over the whole length. As pens 1, 2, 11, and 12 have only $\frac{3}{4}$ the breath of the others, the loads 12 and 11, 12 are each $\frac{3}{4}$, and 2, 3, and 10, 11 are each $\frac{7}{8}$, of

the loads at the other joints. The depth of the girder is $\frac{1}{7}$ of the span. The loads are

$$1,2 = 11,12 = 6 \text{ tons};$$
$$2,3 = 10,11 = 7 \text{ tons};$$
$$3,4 = 4,5 = 5,6 = 6,7 = 7,8 = 8,9 = 9,10 = 8 \text{ tons}.$$

Total load = 82 tons. Each vertical supporting force = 41 tons.

Left-hand cyclic order is chosen, and the load line 1, 12 is, therefore, plotted near the left edge of the paper. The three bars A 1, **A** 12, and **L L** have no stress in them. The two former bars have stresses developed in them when an end wind acts on the bridge; when the girder expands by rise of temperature and thus thrusts horizontally on the abutments; and when the head of a train comes on the bridge, thus communicating to it a horizontal shock. The bar **L L** serves to support part of the weight of the long compression member O (**L L**) and to lessen its buckling by keeping its middle point in line with its end points. None of these three bars are useless, therefore, although the stress-diagram shown indicates zero stress. **L L** must be constructed as a strut, but requires only a small section.

Symmetrical Bridge. Distributed rolling load

The order of the construction is as follows:—

Mark a coincident with 1. Then draw

$${}^{o}_{a}{>}b \quad {}^{b}_{2}{>}c \quad {}^{o}_{c}{>}d \quad {}^{d}_{3}{>}e \quad {}_{e}{>}f \quad {}^{f}_{4}{>}g \quad {}^{o}_{g}{>}h \quad {}^{h}_{5}{>}i \quad {}^{o}_{i}{>}j \quad {}^{j}_{6}{>}k \quad {}^{o}_{k}{>}l.$$

Mark **1** coincident with l. Then ${}^{\mathbf{1}}_{7}{>}\mathbf{k}$, &c. The lower half of the diagram is symmetrical with the upper half; $j\,h$ and f should coincide with the corresponding heavy type letters, and $k\,i\,g\,e\,c\,d\,b$ should be vertically over the corresponding heavy letters. These latter coincidences are the tests of accuracy in drawing.

In the stress-diagram No. 2 the pen 12 is supposed to be unloaded, the other portions being loaded as before. All the

joint loads remain the same except 11, 12, which becomes 3 tons, and the supporting forces 0 1 becoming $41 - \frac{9}{44} = 40\frac{35}{44}$ tons, and $12, 0 = 41 - 2\frac{35}{44} = 38\frac{9}{44}$ tons. The order of proceeding in drawing the diagram is the same as above. This diagram is shown in the plate drawn on top of No. 1 diagram, and distinguished by being drawn in dotted lines, thus - - - - -.

The third diagram, drawn in same place with —·—·— lines, is for the load covering all but the last two pens 11 and 12. The load 11, 12 is now zero, and $10, 11 = 4$ tons, the others remaining unchanged. The total load is now $82 - 9 = 73$ tons, of which $40\frac{35}{44} - \frac{27}{44} = 40\frac{2}{11}$ tons is balanced by 0 1, and the remaining $32\frac{9}{11}$ tons by 12, 0.

The fourth diagram shows the results with all but the last three pens loaded as before; the fifth, those with all but the last four pens loaded; and so on, there being 12 stress-diagrams drawn, the last of which corresponds to the case of the first pen only being covered with a load of 6 tons, and the only external forces considered being, therefore, a load of 3 tons in line 1, 2, and the supporting forces, which are, in this case, $0\,1 = 3 \times \frac{41}{44} = 2\frac{35}{44}$ tons, and $12, 0 = 3 \times \frac{3}{44} = \frac{9}{44}$ ton. The student should carefully trace through all twelve diagrams and compare the results, which are very instructive. It will be noticed that the maximum stress in any bar of the upper or lower chord occurs when the bridge is fully loaded; while the maximum for any diagonal or vertical bracing bar occurs when the larger only of the two portions of the bridge to one side of the lower end of the bar is covered by the load. This is not an invariable law, as will be seen presently, but it commonly holds good for the shapes employed in engineering structures. As under these various systems of loadings the individual links are sometimes in tension and sometimes in compression, it is impossible to distinguish between struts and ties by thick and thin lines in the frame-diagram. All those bars that are at any time in compression have been marked by thick lines; those that are always in tension by thin lines.

Symmetrical Bridge. Distributed rolling load

51. The 'Method of Sections' has already been mentioned. One of its uses is to investigate the stresses in part of a structure without taking the trouble to draw out a stress-diagram for the whole. The dotted-line section P Q in Fig. 83 may be taken as an example. It crosses the pens 2, C, D, and 0, two of which are outside and two inside. The method is applicable only to a section crossing not more than 2 inside pens—i.e. cutting not more than 3 inside links. The locor sum of all the outside forces (in this case $0\,1 + 1\,2 \mathrel{\#} 0\,2 \mathrel{\#} R$) is first found either by a one-pen linkage, or, as in the present very simple example, by numerical calculation. From the joint between R and the link 2 C is drawn a line to the joint between C D and D 0. The portion of the frame to the left hand of the section line is in balance under the four forces R, 2 C, C D, and D 0. The locor sum of the first two must pass through their joint; the locor sum of the last two must pass through *their* joint; these two locor sums must be equal and opposite along the same line—that is, $R + 2\,C \mathrel{+\!\!\!\!\equiv} C\,D + D\,0$; and therefore each of them lies along the line last drawn. Parallel to this line, therefore, is drawn the line $c\,0$ in Fig. 84, where $0\,2 \mathrel{\#} R$ and $2\,c \parallel 2\,C$.

Method of Sections

This gives $2\,c \mathrel{+\!\!\!\!=}$ stress in link 2 C, and $c\,0 \mathrel{+\!\!\!\!=}$ vector sum of stresses in C D and D 0. This latter is now resolved into its two components by drawing $c\,d \parallel C\,D$ and $0\,d \parallel 0\,D$—that is, by the operation ${}^{c}_{0}\!>\!d$. We have thus obtained the stresses in the three links cut by the section plane P Q. The construction of Fig. 84 may be made in either of two forms as shown—one corresponding to the left-hand cyclic order, in which the locors are cut in making left-handedly the circuit of the portion of the frame on the one side of the section, and the other being similarly right-handed. The former is coincident in every respect with the diagram $0\,2\,c\,d\,0$ of the first stress-diagram of Fig. 83, except that it is drawn to a larger scale.

52. For symmetrical loading the utility of this 'method of sections' is not great, because in this case one does not need to draw a one-pen linkage in order to determine the supporting forces; and, therefore, it must be drawn specially for the sake of finding the position of the locor R in solving the 'section.' But one can draw the complete stress-diagram up to the section-line practically as quickly as the one-pen linkage can be drawn. But for unsymmetrical loading the supporting forces cannot be found without drawing the one-pen linkage, and a portion of this already drawn can be used to fix the position of R. Its use in starting the diagrams of peculiarly formed structures has already been mentioned.

Symmetrical and unsymmetrical Loading

53. Suppose the two boundary links cut by the section-line be parallel to each other and R parallel to both. The joint of R with either of these links is at infinity, and the above construction fails. But in this case evidently the transverse bracing bar has no stress in it, because the resultant external force has no transverse component to be balanced by the thrust or pull of this bar across the plane of section. The stresses in the two boundary links will, therefore, be found by dividing R between them in proportion to their distance from R. This, of course, is easily done in the ordinary graphic mode. This case arises in taking sections of vertical and non-tapered piers with vertical loading.

Parallel links in section

54. This 'method of sections' is not applicable to a section cutting three links which meet at one joint. As an example take Fig. 85. This is not put forward as an economical form of roof truss; it is inserted here simply as illustrating several of the special features that have been spoken of. It contains no joint where only two links meet, and yet it is non-redundant and stiff. A section through the pens 2 B A 0 is not amenable to treatment by the above 'method of sections,' because the three bars 2 B, B A, and A 0 all meet in one joint. The stress-diagram can, however, be commenced by help of

Roof illustrating special difficulties

a section through the pens 4 E F 0, or of one through 5 G F 0. The former is used in the following construction.

The load data are the following. The loads 1 2, 2 3, and 3 4 are vertical, the first 5 tons and 2 3 = 3 4 = $7\frac{1}{2}$ tons. The load 4 5 = 9 tons and is inclined at 30° to the vertical. The load 5 6 = 6 7 = 10 tons, each being inclined at 20° to the vertical, while 7 8 = 7 tons and is at 15° to the vertical. The horizontal component of the supporting force 8 0 is known to be $\frac{1}{3}$ of the horizontal component of 0 1, both components having the same direction (viz. of course, from left towards right hand).

Roof illustrating special difficulties

With these data proceed as follows. Plot out the outside stresses 1 2, 2 3 . . . . 7 8 in consecutive right-hand cyclic order. Join 8 1 of stress-diagram, and parallel to it draw two lines from the supporting points P and P_1. Proceed first on the supposition that the supporting forces are parallel, and, therefore, both parallel to 8 1. Choose any pole p, and draw in a corresponding one-pen polygon, drawing first the links through the spaces 1 2 3 4 5 6 7 8, and carrying the first and last of these up to intersect with the lines drawn through P and P_1 parallel to 8 1. Join these two intersections. This gives the 'closing line' of the pen, and parallel to it draw from p a line intersecting 8 1 in 0′. The actual supporting forces are 8 0′ and 0′ 1 compounded with a pair of equal and opposite forces along the line P P_1. Therefore, from 0′ is drawn a line parallel to P P_1, and 0 is known to lie on this line. The horizontal projection of 1 8 is divided in the proportion of 3 to 1, and 0 is found vertically over the point $\frac{1}{4}$ of this horizontal projection from 8. The supporting forces 8 0 and 0 1 are now known, and lines parallel to them through P and P_1 are to be drawn. That through P meets the link of the one-pen linkage through space 1 in a point from which must now be drawn a line $\parallel p$ 0. The joint of this line with that through pen 4 is to be found. It is called R in the diagram. Through this is drawn a line Q R $\parallel$ 0 4, and the joint Q of this with link F 0 is

found. This joint Q is joined with joint (F E 4 5 G), and parallel to this line is drawn $4f$ in the stress-diagram, f being the joint of this and $of \parallel$ O F. The correct position of f being thus found, the diagram proceeds in the following order:—

$$ {}^{f}_{4}{>}e \quad {}^{f}_{o}{>}a \quad {}^{e}_{a}{>}d \quad {}^{3}_{d}{>}c \quad {}^{2}_{c}{>}b. $$

The line between b and a must now be found parallel to B A, and, therefore, coincident with da, if the drawing has been accurate. Continuing ${}^{f}_{5}{>}g \;\; {}^{o}_{f}{>}l \;\; {}^{l}_{g}{>}h \;\; {}^{h}_{6}{>}i \;\; {}^{i}_{7}{>}k$. A second test of accuracy is now found in that the line joining the points l and k so found should be parallel to L K.

It may be noticed that since 2 B and 3 C are in the same line, we might have commenced by finding bc. Thus draw through points 2 and 3 lines $2b$ and $3c$ parallel to 2 B and 3 C; *any* line such as bc drawn between these two and parallel to B C gives the stress on B C. Having found this, cd can be found by drawing $bd \parallel$ B A and $cd \parallel$ C D, because B A and A D are in the same line. In a similar manner ed may next be found. But this beginning will carry us no further than this, because we now find at each of the joints (A D E F) and (0 1 2 B A) that three unknown forces remain to be found. A similar remark might be made regarding the other side of the roof.

Roof illustrating special difficulties

It should also be observed that, in spite of the dissymmetry of the load distribution, the stresses in A O and O L are equal, as are also those in A F and F L. This results from the symmetry of the structure and from the fact that, while the joints (O A F) and (O F L) are 3-link only and are directly joined by a link O F, no external forces act at these. They would still have been equal in pairs if equal and symmetrically disposed loads acted upon these joints; for instance, equal portions of the weights of the bars themselves form actually occurring loads which still leave the above equalities undisturbed.

If the links O A, O F, and O L were in line (along PP_1), then A F and F L would be useless, there being no stress in them. Their only utility would be in suspending the long tie-bar (PP_1) and preventing it drooping by its own weight.

Two trials and errors

55. The other method previously referred to as an alternative to the method of sections in starting a diagram, when there exist no 2-link joints, is the 'Method of Two Trials and Two Errors.' In the present example one would *assume* a value of *o a*, and from it find the stresses in the neighbouring bars—namely, the following stresses: 2 *b*, *a b*, 3 *c*, *b c*, *c d*, *a d*, *d e*, *e f*. The point *e* having now been determined, a line from it parallel to E 4 should cut the load line at point 4 in the stress-diagram. It does not do so, however, because of the error in the assumption made regarding *o a*. The distance of the intersection from 4 is a measure of the error made. Next assume a new value of *o a* and repeat the process, obtaining a new error or deviation from the point 4. By linear interpolation between these two errors the true points are at once obtained. It does not matter in the least whether the two guesses are near or far away from the truth. If the two errors come out + and −, so much the better; and if one error be small and the other large a good accurate interpolation will be obtained.

Cantilever Bridge

This process is illustrated in detail in Fig. 86, in which is represented a bridge structure without any 2-link joints. The frame is of the cantilever type which has become common in recent years.

One way in which this problem might be commenced would be to take a section through the joint at the top of pen A and through the bar O A. The moment of the force in O A round the above joint must balance the moment round the same joint of the resultant of the outside forces 0 1 . . . . . 8 9. The locor sum of these would be found, and its intersection with O A joined with the joint A B U. This joining line gives

the direction of the third side of a stress-triangle of which one side is the stress in O A.

A second mode of commencing would be to take a section through pens 9, U, B, O, or one through 10, **U**, **B**, O, proceeding as previously explained.

In either of these methods the intersection of the locor-sum of external forces with the line of the bar O A or O B would be an ill-conditioned intersection, and although either solution would probably be more rapid than the following, neither would be so accurate.

The load data are the following :—

Cantilever Bridge. Two trials and errors

The downward loads are applied to the joints along the horizontal straight line R Q, R M, S L, S I, &c., all which loads are vertical, and are transferred to the upper boundary by the introduction of imaginary links; and the two supporting forces are applied at joints (0 1 H G) and (18, 0 **G** **H**), that at the latter joint being known to be inclined at 30° to the vertical, and the direction of the former having to be found by help of a one-pen frame. (This latter is shown in the figure in lines thus —·—·—·—, and its stress-pole is marked ⊕.) The loads at the joints

(Q 12), (P Q R), (L M R), (K L S) = each 13 tons.

Those at joints

(F I S), (E F T), (C D T), (B C U) = each 20 tons.

Those at joints

(A B U), (**C B U**), (**D C U**), (**E D T**), (**I F T**), (**K I S**) = each 25 tons.

Those at joints

(**M L S**), (**N M R**), (**Q R**, 17) = each 12 tons.

Total downward load = 318 tons.

Using the method of two trials and errors, we may begin (after having drawn in the closed polygon of outside forces including 18, 0, and 0 1) by choosing for 'trial' any arbitrary

value $1\,q'$ for the stress in 1 Q. From this point q' chosen arbitrarily on the line $1\,q$ drawn from $1 \parallel 1$ Q, we proceed to find the following other points, in which the accent $'$ indicates that their positions correspond with the supposition that q' is the correct position of q :—

$$\substack{q'\\2}\!>r_1' \quad \substack{r_1'\\3}\!>r_2' \quad \substack{r_2'\\4}\!>s_1' \quad \substack{s_1'\\5}\!>s_2' \quad \substack{1\\q'}\!>p' \quad \substack{1\\p'}\!>n' \quad \substack{n'\\r_2'}\!>m'$$

$$\substack{m'\\s_1'}\!>l' \quad \substack{1\\l'}\!>k' \quad \substack{1\\k'}\!>h' \quad \substack{0\\h'}\!>g' \quad \substack{0\\g'}\!>e' \quad \substack{k'\\s_2'}\!>i' \quad \substack{e'\\i'}\!>f'$$

$$\substack{f'\\s_2'}\!>t_1' \quad \substack{t_1'\\6}\!>u_1' \quad \substack{u_1'\\7}\!>u_2' \quad \substack{u_2'\\8}\!>u_3' \quad \substack{u_3'\\9}\!>u_4' \quad \substack{t_1'\\u_2'}\!>t_2' \quad \substack{e'\\t_2'}\!>d'$$

$$\substack{d'\\u_3'}\!>c' \quad \substack{c'\\u_4'}\!>b'.$$

Having now found the point b', we notice that it is not on the line $o\,b$ drawn from $o \parallel$ O B as it ought to be.

A second trial value of $1\,q$—viz. $1\,q''$—is now taken, and exactly the same process as above repeated, resulting in a second wrong position for b, namely b'', which is still out of

Cantilever Bridge. Two trials and errors

the line $o\,b$ but not at the same distance from it as b'. Next through b' and b'' a straight line is drawn to cut $o\,b \parallel$ O B. The intersection b is the true position of b, and the true diagram can now be drawn backwards from b towards q by reversing the series of operations indicated above; thus,

$$\substack{9\\b}\!>u_4 \quad \substack{8\\u_4}\!>u_3 \quad \substack{7\\u_3}\!>u_2 \quad \substack{6\\u_2}\!>u_1 \quad \substack{u_3\\b}\!>c.$$

Here we are stopped in this backward course because no proximate joint leaves only two stresses undetermined. This will not be generally the case, this special example having been chosen because it presents an unusual number of peculiar difficulties.

Next draw through q' and q'' two parallel lines in any direction different from $o\,q$, and plot off along these from q' and q'' to ρ' and ρ'' the distances $b'\,b$ and $b''\,b$. These are measures of the errors resulting from the two trials. In this example the errors are of opposite signs, and are therefore

plotted in opposite directions from q' and q''. Join ρ' and ρ'', and let the joining line meet $o\,q$ in q. This intersection q is the correct position of q, and from it the diagram may be proceeded with up to c in the same order as that used for the trials. That this forward series of operations gives the same position for c as the above-mentioned backward series, affords a proof of the accuracy of the drawing. Evidently the backward operations were not really necessary, because, having found q correctly, the forward series could be continued as in the trials up to b. The proof of accuracy of drawing would then consist in finding the b given to lie on $o\,b \parallel$ O B and also in the line $b'\,b''$.

The rest of the diagram may now be drawn as follows :—

$$\genfrac{}{}{0pt}{}{o}{b}\!>a \quad \genfrac{}{}{0pt}{}{o}{a}\!>\mathrm{b} \quad \genfrac{}{}{0pt}{}{\mathrm{b}}{10}\!>\mathrm{u}_4 \quad \genfrac{}{}{0pt}{}{\mathrm{u}_4}{11}\!>\mathrm{u}_3 \quad \genfrac{}{}{0pt}{}{\mathrm{u}_3}{12}\!>\mathrm{u}_2 \quad \genfrac{}{}{0pt}{}{\mathrm{u}_2}{13}\!>\mathrm{u}_1 \quad \genfrac{}{}{0pt}{}{\mathrm{b}}{\mathrm{u}_3}\!>\mathrm{c}.$$

Cantilever Bridge. Two trials and errors

At this point the same difficulty reappears; there is no proximate joint at which only two as yet undetermined stresses act. Since **q** is not yet known, we have once more to fall back on the method of trial and error.

Draw from $\mathbf{u}_2$ a line $\parallel \mathbf{U}_2\,\mathbf{T}_2$ and select at random any point $\mathbf{t}_2'$ upon it. Then proceed as follows :—

$$\genfrac{}{}{0pt}{}{\mathrm{t}_2'}{\mathrm{u}_1}\!>\mathrm{t}_1' \quad \genfrac{}{}{0pt}{}{\mathrm{c}}{\mathrm{t}_2'}\!>\mathrm{d}' \quad \genfrac{}{}{0pt}{}{o}{\mathrm{d}'}\!>\mathrm{e}' \quad \genfrac{}{}{0pt}{}{o}{\mathrm{e}'}\!>\mathrm{g}' \quad \genfrac{}{}{0pt}{}{18}{\mathrm{g}'}\!>\mathrm{h}' \quad \genfrac{}{}{0pt}{}{18}{\mathrm{h}'}\!>\mathrm{k}' \quad \genfrac{}{}{0pt}{}{\mathrm{e}'}{\mathrm{t}_1'}\!>\mathrm{f}'$$

$$\genfrac{}{}{0pt}{}{\mathrm{k}'}{\mathrm{f}'}\!>\mathrm{i}' \quad \genfrac{}{}{0pt}{}{\mathrm{i}'}{\mathrm{t}_1'}\!>\mathrm{s}_2'.$$

This $\mathbf{s}_2'$ is found not to be on the line 14 s drawn from 14 $\parallel$ 14 S, as it ought to be. Another point $\mathbf{t}_2''$ is now selected on $\mathbf{u}_2\,\mathbf{t}_2$, and the process repeated whereby a point $\mathbf{s}_2''$ is found, still not on the proper line 14 **s** but not at the same distance from it as $\mathbf{s}_2'$. Then $\mathbf{s}_2'\,\mathbf{s}_2''$ are joined and the line produced to meet 14 **s**, the intersection $\mathbf{s}_2$ being the true position of $\mathbf{s}_2$. The two errors $\mathbf{s}_2'\,\mathbf{s}_2$ and $\mathbf{s}_2''\,\mathbf{s}_2$ are now plotted from $\mathbf{t}_2'$ and $\mathbf{t}_2''$ along parallel lines in any direction to the points π' and π''. The ine joining π' and π'' cuts $\mathbf{u}_2$ **t** in $\mathbf{t}_2$, the correct position of $\mathbf{t}_2$.

This point having been obtained, the drawing of the true diagram proceeds as in the trials up to the point $\mathbf{s}_2$. It is then drawn as follows :—

$$\begin{matrix}\mathbf{s}_2\\15\end{matrix}\!> \mathbf{s}_1 \quad \begin{matrix}\mathbf{s}_1\\16\end{matrix}\!> \mathbf{r}_2 \quad \begin{matrix}\mathbf{r}_2\\17\end{matrix}\!> \mathbf{r}_1. \quad \begin{matrix}\mathbf{r}_1\\18\end{matrix}\!> \mathbf{q} \quad \begin{matrix}\mathbf{q}\\18\end{matrix}\!> \mathbf{p} \quad \begin{matrix}\mathbf{p}\\18\end{matrix}\!> \mathbf{n}$$

$$\begin{matrix}\mathbf{k}\\\mathbf{s}_1\end{matrix}\!> \mathbf{l} \quad \begin{matrix}\mathbf{l}\\\mathbf{r}_2\end{matrix}\!> \mathbf{m}.$$

Finally the points **m** and **n** are joined, and the final test of accuracy of drawing is that this line **m n** should be parallel to **M N**.

Cantilever Bridge

It is to be noticed that the diagram shows a stress $r_2\, s_1$ different from the outside force 3 4, which is really applied at joint (L M R S). Now of this stress $r_2\, s_1$ a part = 3 4 belongs to the imaginary link supposed to lie along this line. The stress on the actual link R S is, therefore, the *algebraic* difference between $r_2\, s_1$ and 3 4, the stress in the imaginary link being, of course, a compression. Since $r_2\, s_1$ indicates a resultant *tension*, the real tension on the actual link is the *arithmetic sum* ($r_2\, s_1$ + 3 4), taking both these lengths positively. This is a most important rule to attend to in using imaginary links; if it be not very carefully adhered to, in the case of imaginary and actual links lying along the same line, the stress-diagram will be altogether wrongly interpreted as regards these actual links. The same remark applies in the present example to the links S T, **T S**, and **S R**. The stresses on these four actual links are

on R S	+ 65 + 13 = + 78	tons	tension;
,, S T	− 355 + 20 = − 335	,,	compression;
,, **T S**	− 343 + 25 = − 318	,,	compression;
,, **S R**	+ 61 + 12 = + 73	,,	tension.

Here the imaginary stresses 13, 20, 25, and 12 are added (i.e. given a + sign), because they are compressions (or negative stresses) to be subtracted.

CHAPTER XI.

FLAT STATIC STRUCTURES CONTAINING BEAM-LINKS.

1. WHEN the structure contains beam-links—that is, links exposed to and capable of resisting bending moments—it is necessary to determine these bending moments for all the sections, as well as the shear force and the direct pull or thrust along the axis of the beam. The total or resultant force across any section is always capable of being looked on as a single force through the centre of figure of the section combined with a force-couple. The latter furnishes the bending moment on the section. The former is usually resolved into two components: one, viz. the shear force, parallel to the section plane, and the other normal to it and producing either compression or tension.

Beam stresses

2. There are various methods by which these beam stresses may be dealt with graphically. One is to suppose substituted for the beam a stiff non-redundant triangulated frame. When properly interpreted the stress-diagram for this substituted frame gives all the required forces acting on the actual beam. This method, although legitimate and elegant, is seldom *necessarily* followed. A somewhat similar device was employed in his paper on 'Frictional Efficiency of Mechanisms,' read by Professor Fleeming Jenkin before the Royal Society of Edinburgh in April 1877 and February 1878. Another mode of solution is the successive use of the 'method of sections' at various parts of the frame; or else of the method of 'two trials and two errors.'

Various methods

3. In some simple cases, however, it is not necessary to

resort either to this substitution or to the latter method. The diagram on account of its simplicity becomes easily intelligible without the artificial aid of the supposititious triangulation. Fig. 87 illustrates the method to be adopted for such simple linkages. The beam is indicated by a double line—viz. a fine line indicating the position of its axis, underneath which is drawn a thick line.

The loads are given in direction and magnitude. The supporting force 8, 9 is given in direction; it is vertical. By the construction of a single-pen linkage the direction of 14, 1 and the magnitudes of 14, 1 and 8, 9 are found. Beginning at the joint 4 5 C, the stress-diagram is then constructed as follows:—

$$\genfrac{}{}{0pt}{}{4}{5}{>}c \quad \genfrac{}{}{0pt}{}{3}{c}{>}b \quad \genfrac{}{}{0pt}{}{2}{b}{>}a \quad \genfrac{}{}{0pt}{}{c}{6}{>}d \quad \genfrac{}{}{0pt}{}{d}{7}{>}e.$$

This supplies the poles of all the inside pens. The points *a b d e* are then joined with 14, 13, 10, 9; and *c* with both 12 and 11. Next dotted lines from *a b c d e* are drawn parallel to the beam axis, and their intersections with the load line 9, 14 are marked *a′ b′ c′ d′ e′*.

Simple cases

The stresses on the non-beam links require no comment. The force actions of the various parts of the beam, exclusive of couples or pure bending moments, on the pins of the different joints are as follows: That of part

A, 14	on	pin	14, 1, 2 A	is	*a* 14	and	on	pin	14 A B 13	is 14 *a*;
B 13	,,	,,	14 A B 13	,,	*b* 13	,,	,,	,,	13 B C 12	,, 13 *b*;
C 12	,,	,,	13 B C 12	,,	*c* 12	,,	,,	,,	12, C, 11,	,, 12 *c*;
C 11	,,	,,	12 C 11	,,	*c* 11	,,	,,	,,	11 C D 10	,, 11 *c*;
D 10	,,	,,	11 C D 10	,,	*d* 10	,,	,,	,,	10 D E 9	,, 10 *d*;
E 9	,,	,,	10 D E 9	,,	*e* 9	,,	,,	,,	9 E 7 8	,, 9 *e*.

Each of these except the first and last is combined with a bending moment. Each of them is conveniently resolved by the dotted lines into a shear force normal to the beam-axis (which direction here coincides with that of the lower external

loads) and an axial force producing direct tension or compression.

For instance, the force 13 b with which the part 13 B acts on the joint (13 B C 12) is resolved into the upward shear 13 b' and the axial pull $b'\,b$. The force of part 10 D on joint (10 D E 9) is resolved into the downward shear 10 d' and the pull $d'\,d$. With the particular data of this example the beam is in tension throughout its whole length, but in general it may be in tension at one part and compression at another. At the joint of the force-line 11, 12 with the beam—viz. joint (11, 12, C)—the shear force of the left-hand portion is 12 c', and that of the right-hand portion is c' 11, the shear in both these portions being of the same sign—viz. such as to give a right-handed twist to the material of the beam viewed as in the figure. The sign of the shear changes at joint (11 C D 10), the actions 11 c' and d' 10 on the two sides of this joint being both upwards.

Simple cases

A B and B C are tie-bars; C D and D E are struts, as also 2 A and E 7. The forces acting on the beam are those exerted by these bars and the external loads applied at the beam-joints. The components of these forces parallel to the beam-axis have no bending moments on the sections. To make this statement generally true, it must be carefully noted that the joints where the forces are resolved into their components must be taken as the *geometrical* joints between the force-lines and the beam-axis. The actual joint-pin centres—that is, the actual points of application of the forces to the beam—may lie out of the beam-axis, in which case the draughtsman must be careful to avoid using these actual joints instead of the geometrical joints of the axial lines. At the right-hand end of the beam the forces e 7, 7 8, and 8 9 have the resultant e 9, whose vertical component is e' 9. At the left-hand end the forces are 14, 1 + 12 + 2 a $\#$ 14 a, whose vertical component is 14 a'. The vertical component of $a\,b$, the force exerted by link A B on the beam, is $a'\,b'$; and so on. The

normal, or bending-moment-producing, forces acting on the beam are, therefore,

$$e'\,9\,;\ 9,\,10\,;\ 10,\,11\,;\ 11,\,12\,;\ 12,\,13\,;\ 13,\,14\,;\ 14\,a'\,;$$
$$a'\,b'\,;\ b'\,c'\,;\ c'\,d'\,;\ d'\,e'.$$

We find these arranged in consecutive order in the stress-diagram—that is, forming a continuous and closed chain. The pole p is chosen, making the pole-distance $p\,(e'\,9)$ a convenient multiplier. In this case twenty tons is used. Then parallel to the pencil (p) e' 9, 10, 11, 12, 13, 14, $a'\,b'\,c'\,d'$ is drawn the one-pen linkage (P) E′ 9, 10, 11, 12, 13, 14 A′ B′ C′ D′. It is to be observed that the spaces A′ B′ C′ D′ E′ do not coincide with A B C D E, the dividing lines A′B′, B′C′, &c., being vertical—i.e. normal to the beam-axis. If the work has been accurate this will be a closed linkage. Its vertical depth directly under any section multiplied by the pole-distance $p\,(e'\,9)$ gives the bending moment on that section. Since the frame has been drawn to the scale $\frac{1}{5}'' = 1$ ft., therefore, one inch depth on moment-diagram $= 5$ ft. $\times$ 20 tons $= 100$ foot-tons.

Simple cases

We have thus obtained a diagram giving the stresses on all the two-joint links, and also the direct thrust or pull, the shear force, and the bending moment on every section of the beam.

In Fig. 88, a structure of similar design, but with the loads on the lower joints oblique to the beam axis, is shown. The points $a\,b\,c\,d\,e$ are found as in last case. Then from 14 is drawn a line 14 a' perpendicular to the beam axis, and upon it are projected at right angles to it the points $a\,b\,c\,d\,e$ 9, 10, 11, 12, 13. The projections are marked $a'\,b'\,c'\,d'\,e'$ 9′ 10′ 11′ 12′ 13′. Then 14 a', 13′ b', 12′ c', 11′ c', 10′ d', 9′ e' are the shears on the transverse sections of the portions 14 A, 13 B, 12 C, 11 C, 10 D, 9 E of the beam. The shear changes sign at joint (13 B C 12). The axial thrust

along 14 A is $a' a$; and that along 13 B is $(b b' - 13, 13')$. The rest of the beam is in tension, the tensive stresses being $(12, 12' + c' c)$ on 12 C; $(11, 11' + c' c)$ on 11 C; $(10, 10' + d' d)$ on 10 D; and $(9 9' + e' e)$ on 9 E.

The set of transverse forces producing pure bending are: **Simple cases**

$$14 a', a' b', b' c', c' d', d' e', e' 9', 9' 10', 10' 11', 11' 12', 12' 13', 13' 14'.$$

These being arranged consecutively along the line $14 a'$, a pole p at a suitable pole-distance is chosen, and from it the moment diagram or one-pen linkage (P) 14′ A′ B′ C′ D′ E′ 9′ 10′, &c., is drawn through the spaces divided by the dotted lines drawn from the beam-joints normal to its axis.

4. In these cases the easiness of the problem arises from the fact that there exists in the frame a joint at which only two links meet, these two links being each a two-joint link— i.e. not a beam. The diagram can be at once begun at this joint.

In Fig. 89 at the right and left hand extremities there are joints where only two links meet, but in each case one of the two links is a beam. The reaction of the beam on the pin is not in the direction of the beam axis; it is in an unknown direction, and, therefore, the diagram cannot be commenced at either of these points.

A section is taken through pens 4 B 1. The resultant locor R ⧺ 1 2 + 2 3 + 3 4 is found and its joint with B 1 determined. This joint is joined with joint (B 4 5 C), and parallel to the joining line is drawn $4 b$ in the stress-diagram to meet $1 b \parallel 1$ B at the point b. $b 4$ gives in direction and magnitude the force exerted by the beam on the joint-pin B 4 5 C; also $b 1$ gives the true stress on B 1. The diagram now proceeds as follows:— **Roof with four beams**

$\substack{b \\ 1} > a$; join $a 3$; take section through spaces 1E and joint (E D 6 7 F); find resultant locor R′ ⧺ (1 2 + 2 3 + 3 4 + 4 5 + 5 6) and its joint with E 1; join this joint with joint (E D 6 7 F),

and parallel to the joining line draw $6e$ in stress-diagram to meet $1e$ drawn $\parallel$ 1 E; this gives e; then ${}^{e}_{b}\!>c$; ${}^{e}_{c}\!>d$; join $c5$ and $d6$; take section through spaces 9 H 1; find R″ $\#$ (9, 10 + 10, 11 + 11, 1) and its joint with H 1; join this joint with (H G 8 9) and parallel to the joining line draw $9h$ in stress-diagram to meet $1h \parallel$ 1 H; then ${}^{1}_{h}\!>j$; ${}^{e}_{h}\!>g$; ${}^{e}_{g}\!>f$; join $g8$ and $f7$.

The beam (3, 4) (A, B) is kept in balance by the following forces:—

$$12,\ 23,\ 34,\ 45,\ 5c,\ cb,\ ba,\ a1.$$

These resolved transversely across the beam are:—

$$1'2',\ 2'3',\ 3'4',\ 4'5',\ 5'c,\ cb,\ ba,\ a1',$$

Roof with four beams

where it must be noted that the struts A B, B C, C D are all normal to the beam-axis. The shears upon the two parts 3 A and 4 B of the beam are $3'a$ and $4'b$, these being of opposite signs. The axial stresses are $33'$ on 3 A and $44'$ on 4 B, both being thrusts. The bending moment-diagram is drawn for the above set of balancing transverse forces—namely, $a3'$, $3'4'$, $4'b$, and ba, the three forces $a1'$, $1'2'$, $2'3'$ being equal to the single force $a3'$, and similarly for the other end. The pole-distance p_1 $(3'b)$ is chosen so as to give a convenient scale for the moment-diagram. The force-diagram having been drawn to the scale $\frac{1}{4}$ inch = 1 ton, the pole-distance has been taken 2 tons. The moment-diagram consists of four lines only, and is called P_1 in the truss-diagram.

The beam (5, 6) (C, D) is treated in the same way. The shears are $5'c$ and $6'd$ in the two portions of the length, and the axial forces, both thrusts, are $55'$ and $66'$. The moment-diagram is drawn with the pole p_2 with same pole-distance 2 tons from the pencil (p_2) $c56d$, and is marked P_2 in the diagram.

The beams (7 8) (F G) and (9, 10) (H J) are treated simi-

larly, poles p_3 and p_4 being chosen for them with the same pole-distance—namely, 2 tons—as for the two beams at the left-hand side of the the truss.

Here the substitution of triangulated frames for the beams has been avoided by making free use of the 'method of sections,' no less than three different sections having been requisite for the completion of the diagram.

5. In Fig. 90 is shown a simple illustration of common occurrence of a beam structure which can be dealt with by the method of sections. It is a pier composed of two upright columns capable of resisting bending and braced together by three two-joint links. The upper joints are exposed to vertical loads, and horizontal wind forces act on the two joints of one column. The structure is supported at P_1 and P_2, where it is pin-jointed to the foundation. The force exerted by the foundation at P_1 is known to be vertical; that at P_2 (or 6 1) is found by the construction of a single-pen linkage, the pole p being used. Then the line 6 1 in the frame is produced to meet 1 B, and its joint with this line is joined with joint (6 B A 5). Parallel to this last line is drawn 6 b in the stress-diagram to meet in b the line 1 b drawn parallel to 1 B. This gives the true position of b as is recognised by considering a section through spaces 1 and B and through joint (6 B A 5). **Braced Pier**

Then proceed with ${}^{b}_{5}\!>\!a$. In the diagram it is an accident that the point a falls on the line 6 2. Next join a 3 and project a vertically to a' on line 2 4, and project 6 to 6′ on 1 b. The axial and shear stresses on the beams are now:

	Axial	Transverse or shear
Lower part of left-hand column . .	12 tension	0
Upper part of left-hand column . .	$a\,a'$ compression	$3\,a' = 0$
Lower part of right-hand column . .	6 6′ compression	1 6′
Upper part of right-hand column . .	6 6′ compression	b 6′

The transverse forces on the left-hand column are 2 3, and the horizontal component of a 1, applied at joint (A B 1 2 3), and (3 4 + 5 a), applied at joint (3 4 5 A). Since these combine into zero forces applied at these points, the moment-diagram reduces to zero, which means that the stress along this column is wholly axial.

Braced Pier

The transverse bending forces on the right-hand column are b 6′ applied at the top joint, and which is the horizontal component of the resultant ($b\,a + a\,5 + 5\,6$); 6′ 1 applied at the bottom joint, and 1 b applied at the middle joint. The pole distance p_1 (1 b) is chosen equal to 10 tons, and from the pencil (p_1) b 6′ 1 is drawn the moment-diagram P. The scale of this diagram is $\frac{1}{50}$ inch = 1 foot-ton.

The left-hand column is a pure strut, not exposed to transverse bending forces. This is the result of our having taken as one of the *data* that the force at P_1 is vertical—that is, in the direction of the axis of the column. If it had been in any other direction there would have been a moment-diagram for this column as well as one for the other.

Roof with three beams. Trial and error

6. In the example taken in Fig. 89 it was possible to start by taking a section. The section was through one tension-bar and a beam; but the part of the beam cut by it was an *end* part—i.e. no joint intervened in the beam between the section and the end of the beam. Thus a point was known—namely, the end joint of the beam—through which the resultant force-action of this part of the beam was known to act. It was only for this reason that the section was directly soluble, and in fact it might be simpler to consider the section as taken through the end beam-joint and the tension-bar. A section through a two-joint link and a beam joint which is not an end joint (and at which, therefore, the bending moment is not necessarily zero) is not directly soluble. In the following example (Fig. 91) many such sections could be taken, but they are useless as aids in commencing the diagram. There is also

no section possible through a beam end-joint and a simple non-beam link.

The substitution of a triangulated truss for the main lower beam would furnish no direct means of commencing the diagram. If such triangulated trusses were substituted for all the three beams, then the diagram could be wholly deduced by the method of trial and error already explained. This method of trial and error is the only available one for solving this problem, but we will here carry it out without the aid of substituted triangulated frames.

At first sight it might be supposed that this was a peculiarly easy case to deal with because there are no less than three joints—namely (L 4 5), (16 V 15), and (Q 9, 10), at each of which only two links meet; and there are other two—namely, (4 D B 3) and (17 J H 16), where two links are in line and where, therefore, the stress in the third can be found at once. But a little examination shows that the beginnings that can be made at these points do not lead far towards the building up of the diagram.

Roof with three beams. Trial and error

Not counting beam-joints intermediate between the ends, the whole number of joints in the structure is 22. There should, therefore, be $44 - 3 = 41$ links; and this is the actual number, namely, 3 beams and 38 two-joint links.

All the loads on the right-hand side are vertical; those on the left-hand and the top joint are inclined 15° from the vertical, except 12 and 45, which are at 45°. The loads 11, 12; 12, 13; 13, 14; and 15, 16 are each 3 tons; 5 6, 6 7, 7 8, 8 9, and 9, 10 are each 5 tons; 16, 17 and 17, 18 are each 6 tons; 1 2 and 2 3 and 3 4 are each 10 tons; 18, 19 is 4 tons, and 4 5 is 7 tons. The load V V′ is 2 tons and Q Q′ is 4 tons; these are applied at internal joints, but are transferred to the boundary by introducing the imaginary links V V′ and Q Q′ and imaginary joints at the upper ends of these links. The vertical direction is given for the supporting force 19, 20 as one of the data.

After plotting consecutively the above loads, the single-pen P is drawn with help of the pole p, chosen in any convenient position, and the pencil of rays drawn from p to the corners of the load-line. The resultant $R_{1,19}$ of these loads being found by help of this pen, the given line of 19, 20 is produced to intersect this resultant, and the line from this intersection to the left-hand end joint of the lower beam is that of the supporting force at this end. In this way the magnitudes of 19, 20 and 20, 1 are obtained in the stress-diagram.

We now begin by solving the joint Q 9, 10. As the stress on the imaginary link equals the imaginary external force applied at this joint, the balancing diagram can be drawn omitting these in the first place, thus giving the stresses on the bars Q 9 and 11 Q′. These being obtained, the known imaginary forces at this joint are next introduced in their proper cyclical order, and the diagram for the joint is thus completed. The diagram now proceeds thus:—

Roof with three beams. Trial and error

$$\begin{smallmatrix} q \\ 8 \end{smallmatrix}\!> o \quad \begin{smallmatrix} 12 \\ q' \end{smallmatrix}\!> s \quad \begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\!> l \quad \begin{smallmatrix} l \\ 7 \end{smallmatrix}\!> m \quad \begin{smallmatrix} m \\ o \end{smallmatrix}\!> n \quad \begin{smallmatrix} 15 \\ 16 \end{smallmatrix}\!> v' \quad \begin{smallmatrix} 14 \\ v' \end{smallmatrix}\!> v$$

$$\begin{smallmatrix} 13 \\ v \end{smallmatrix}\!> u \quad \begin{smallmatrix} s \\ u \end{smallmatrix}\!> t.$$

So far no trial and error has been used. But we can proceed no further in this direct manner, and the beams F (L N O) and F (S T V) are still unsolved.

The former of these beams along with the portion of the roof lying above it is acted on by a set of forces—

$$4\,5 + 5\,6 + 6\,7 + 7\,8 + 8\,9 + 9\,10 + 10\,11 + 11\,12 + 12\,s,$$

all of which are known. Their resultant is found in the usual way, its magnitude and direction being given in the *stress*-diagram by 4 *s*, and its line in the *frame*-diagram being obtained by help of the link polygon and being marked R_{4s}. This set of forces is to be balanced by a force S F exerted by the right-hand beam at its upper joint and by two forces F D

and D 4, which may meantime be considered as compounded into a single force F 4 acting through the lower joint of the left-hand beam. The lines of these two forces S F and F 4 must intersect in some point of the line R_{4s}.

Guess this point to be β_1, and parallel to the two lines from β_1 to the two ends of the beam draw in the stress-diagram the two lines $4f_1$ and sf_1 from the two already known points 4 and s. Mark their intersection f_1. This is a position of f resulting from *guess* β_1 on line R_{4s}.

Next find also by the link-polygon method the resultants

$$R_{4,16} \# 4\,5 + 5\,6 + 6\,7 + 7\,8 + 8\,9 + 9\,10 + 10\,11 + 11\,12 + 12\,13 + 13\,14 + 14\,15 + 15\,16,$$

and

$$R_{0,16} \# 0\,8 + 8\,9 + 9\,10 + 10\,11 + 11\,12 + 12\,13 + 13\,14 + 14\,15 + 15\,16.$$

The former $R_{4,16}$ is the sum of the known forces acting on the part resting on the two beams, and has to be balanced by

Roof with three beams. Trial and error

$$16\,F \# 16\,H + H\,F \text{ through joint } (16\,H\,F)$$

and

$$F\,4 \# F\,D + D\,4 \text{ through joint } (F\,D\,4),$$

this last being the same force as already mentioned and *guessed* to be in the direction from (F D 4) to β_1. The other resultant $R_{0,16}$ is the sum of the known forces acting on the part resting on the right-hand beam, and has to be balanced by 16 F, the same force as has just been cited, along with F O, the force exerted by the left-hand beam at its upper joint.

Produce the line (F D 4) β_1 to meet $R_{4,16}$ in α_1, and let the line α_1 (16 H F) meet $R_{0,16}$ in γ_1. If the guess β_1 were correct, then γ_1 would be the point in $R_{0,16}$ where the two forces 16 F and F O would meet. Parallel to the two lines from γ_1 to the two ends of the beam F (S T V) draw in the stress-diagram from the known points 0 and 16 the two lines $0f_1'$ and $16f_1'$ intersecting in f_1'. This is another determination of the point f resulting from the guess β_1; and the discrepancy between this position

and the first found f_1 is a measure of the error involved in this guess.

Next guess a second point β_2 on line R_{48}, and repeating exactly the same process, find points $\alpha_2 \gamma_2 f_2$ and f_2'. The latter two are the two discrepant positions of f resulting from the supposition that β_2 is the correct point, and the distance $f_2 f_2'$ is a measure of the error involved in that supposition. From the properties of the 'parallel displacement of the links of the single pen' or the 'proportional displacement of the joints of the single pen,' explained in Section 12 of Chapter VIII., it follows directly that all possible positions of f found from the first step in this process must lie on the line $f_1 f_2$, while the locus of those found from the second step is $f_1' f_2'$. The intersection of these two lines is, therefore, the true position of f. The position thus found should be checked before proceeding with the diagram by drawing from the three joints at the ends of the beams four lines parallel to $f4$, $f8$, $f0$, $f16$. The first pair should meet in R_{48}; the latter in $R_{0,16}$; and the first and the last of the four should meet in $R_{4,16}$. In the example shown these intersections came right without visible error.

Roof with three beams. Trial and error

Having thus found f, the diagram proceeds as follows:

$$\genfrac{}{}{0pt}{}{f}{4}\!>d \quad \genfrac{}{}{0pt}{}{3}{d}\!>b \quad \genfrac{}{}{0pt}{}{2}{b}\!>a \quad \genfrac{}{}{0pt}{}{b}{d}\!>c \quad \genfrac{}{}{0pt}{}{d}{f}\!>e \quad \genfrac{}{}{0pt}{}{f}{16}\!>h \quad \genfrac{}{}{0pt}{}{h}{17}\!>j \quad \genfrac{}{}{0pt}{}{j}{18}\!>k$$

$$\genfrac{}{}{0pt}{}{j}{h}\!>i \quad \genfrac{}{}{0pt}{}{h}{f}\!>g.$$

All the required points are now found, and all the forces acting at the various joints of the three beams are known. These forces are next resolved as in the last examples into axial and transverse components, and from the latter are drawn in moment-diagrams. The stress-poles chosen for these moment-diagrams are marked $p_1\, p_2$ and p_3. The pole-distance taken for the long lower beam is four times as great as that taken for the two short upper beams, so that its moment-diagram appears to a scale only one-quarter as large as that of the short beam diagrams. The accurate closing of these three moment-dia-

grams furnishes additional tests of the correctness of the drawing.

7. In Fig. 92 is shown a stiff non-redundant structure with two beam links. This might be dealt with by means of a section taken through the two-joint link 1 A and the joint (2 1 A)—namely, the joint between the upright beam and the nearly horizontal beam. The section must not pass through the upright beam, but only through the spaces 1 A 2. The joint is a flexible one, so that the force exerted by the upright beam on the other passes through a known point—namely, this joint. The resultant of the forces A 1 and 1 2 must, therefore, pass through this joint. The lines A 1 and 1 2 being produced to meet, their joint would be joined to joint (2 1 A), and parallel to the joining line would be drawn 2 *a* in the stress-diagram from the point 2 to meet in *a* the line 1 *a* drawn from 1 parallel to 1 A.

Crane with two beams. Substituted triangulated truss

The problem is, however, solved here by the aid of a triangular truss substituted for the upright beam as an example of this latter method.

The data include, besides the dimensions of the structure, the magnitude and direction of the loading force 1 2 applied at the extremity of the jib beam; the position and direction of the supporting force 2 3; and the point of application of the supporting force 3 1. These three being the only external forces, their lines must meet in one point. Producing 1 2 and 2 3 to meet, their joint is joined to joint (3 1 3). This line gives the direction of 3 1, and this being known the triangle in the stress-diagram is completed $^{1}_{2}\!>\!3$.

Any point to the left of the upright beam in space 1 is taken as the position of an imaginary pin, and this joined to the four joints of the beam by the imaginary links 1 G, G F, F E, and E 1. In the diagram these links and their stresses are indicated by dotted lines in order to distinguish them clearly from the real links and actually occurring stresses. It

must be distinctly understood that they are only construction lines; they are not used in the final reading off of the stresses throughout the structure, and in practical designing they might be drawn in pencil only, and rubbed out after inking the other lines.

The stress-diagram is now built up as follows:—

$$\genfrac{}{}{0pt}{}{3}{1}\!>\!e \quad \genfrac{}{}{0pt}{}{2}{e}\!>\!f \quad \genfrac{}{}{0pt}{}{f}{1}\!>\!g \quad \genfrac{}{}{0pt}{}{g}{1}\!>\!a \quad \genfrac{}{}{0pt}{}{a}{1}\!>\!b \quad \genfrac{}{}{0pt}{}{b}{1}\!>\!c \quad \genfrac{}{}{0pt}{}{c}{1}\!>\!d.$$

Then 2 is joined with $a\,b\,c$ and d; $1\,1''$ and $2\,2''$ are drawn $\perp$ $3\,e$; through a is drawn $1'\,d'$ $\perp$ the beam 2 (A B C D), and $1\,1'$, $2\,2'$, $b\,b'$, $c\,c'$ and $d\,d'$ are drawn $\perp$ $1'\,d'$.

All the forces acting on the beam 2 (A B C D) are now found, so that it is unnecessary to suppose substituted for this beam an imaginary triangulated truss.

Crane with two beams. Substituted triangulated truss

The interpretation of the diagram so far as the beam stresses are concerned is the following:—

Upright Beam.

In part	Axial stresses	Shear stresses
3 1 or 3 E	$3\,1''$ compression	$1''\,1$
2 1 or 2 F	$2''\,1''$ compression	Zero
A 1 or A G	Zero	$a\,1$

The axial stress on the top part A 1 of the beam is zero because the two-joint link A 1 is placed at right angles to the beam: if it were inclined otherwise this axial stress would not be zero.

The shear stress is zero in the middle part of this beam because the load 1 2 acts parallelly to this beam, so that the components of 2 3 and 3 1 perpendicular to the beam must balance each other—that is, the horizontal component of 2 3 + 3 1 ⧺ 2 1 is zero. The shears on top and bottom parts are of opposite signs.

Inclined Beam.

In part	Axial stresses	Shear stresses
2 A	$2\,2'$ compression	$2'\,a$
2 B	$(2\,2' - b\,b')$ compression	$2'\,b'$
2 C	$(2\,2' - c\,c')$ compression	$2'\,c'$
2 D	$(2\,2' - d\,d')$ compression	$2'\,d'$

The shear in this beam changes sign at joint (2 C D), and it is here, of course, that the maximum moment occurs. In the upright beam the bending moment is constant throughout the middle part 2 F where the shear is zero.[1] Two poles p_1 and p_2 are chosen at a convenient (and each at the same) distance from the lines $1''a$ and $a\,d'$, and from the pencils (p_1) $1''1\,a\,1''$ and (p_2) $a\,b'\,c'\,d'\,1'2'\,a$ are drawn the two moment-diagrams P_1 and P_2 for the upright and inclined beams.

Crane with two beams. Substituted triangulated truss

The force $a\,1$ which the vertical beam exerts on the pin at its top joint is the resultant $(a\,g + g\,1)$. The force $2\,1$ with which the middle part of this beam acts on the pin at top of middle part is the resultant $(2f + fg + g\,1)$; the stiff beam section is here equivalent in its action to the three two-joint links 2 F, F G, and G 1 of the substituted triangulated truss. On the same pin the upper part of the beam acts with a force $1\,a$, the resultant $(1\,g + g\,a)$. The vertical beam has, therefore, a total force-action on this same pin, and therefore also upon the left-hand end of the other beam, equal to the resultant $(2f + fg + g\,1 + 1\,g + g\,a) \;\#\; 2\,a$. The reaction of the inclined beam on the vertical beam is $a\,2$.

The positions of the points efg depend altogether upon that arbitrarily chosen for the joint (1 G F E); but the actual stresses in the diagram are not dependent in any degree upon the positions of efg. The student should satisfy himself that the positions obtained for the points $a\,b\,c\,d$ do not vary with

[1] The bending moment on a beam always reaches its maximum at the section where the shear is either zero or changes sign.

change of position of joint (1 G F E) by taking several different points for this joint and working out the diagrams for all of them. If the drawing be accurate, the different diagrams will all lead to the same points *a b c d*, and the two moment-diagrams will both close.

CHAPTER XII.

SOLID STATIC STRUCTURES.

1. STRUCTURES built in three dimensions offer naturally much more difficulty in the investigation of their stresses than do plane structures. Notwithstanding this superior difficulty, it is of the greatest engineering importance to be able to deal with them so as to arrive at a more or less exact knowledge of the forces through the various sections. The importance of this part of the subject becomes evident when it is remembered that as a matter of fact *all* structures are solid. A girder or roof-truss may be treated as a plane structure with a certain approximation to truth so long as the forces acting on it are parallel to its central plane; but it must not be forgotten that the girder, viewed in this light, is in unstable equilibrium transversely to its plane, and that it would not stand up when side winds act upon it except for the wind-bracing. This wind-bracing does not lie in the above plane, and when it is included as part of the girder the structure becomes a *solid* one. **All structures are solid**

2. It will be well at the outset to tabulate the fundamental distinctions between plane and solid link structures as regards stiffness, equilibrium, and solubility of the stress problem for an individual joint.

Stiffness.—It has already been explained that to ensure stiffness and avoid redundancy, the number of two-joint links in a plane structure must be three less than double the number of joints; and that in a solid structure it must be six less than thrice the number of joints. **Flat and solid frames compared**

Equilibrium.—In a plane frame, when the known loads are balanced by a single force, as in a steel-yard, the condition of balance determines the magnitude, the direction, and the position of the necessary balancing force.

In a solid frame, similarly, when the loads are balanced by a single force, this is determined in magnitude, direction, and position by the condition of balance; but it is not generally possible to produce the balance by a single force.

When the loads on a plane frame are balanced by two forces, the two points of application and the direction of one of them are independent of the condition of balance; and, these being otherwise fixed, the direction of the second and the magnitudes of both are determined by this condition. A further limitation is that the two supporting forces must be in the plane of the frame and of the loads.

Flat and solid frames compared

If the loads on a solid structure be balanced by two forces, the supporting forces must lie in one plane with the resultant of the loads, and they are subject to the last set of conditions; but, again, it will not in general be possible to obtain balance with two forces only.

If the loads on a solid structure be balanced by three forces, then the three points of application, the direction of one absolutely, and the direction in a given determined plane of a second supporting force, are independent of the balance-condition; but, these being otherwise fixed, this balance-condition determines the magnitudes of all the three supporting forces, the plane of the second, and the direction absolutely of the third—that is, as regards the second, the condition of balance fixes a plane in which it must lie, but not its direction in that plane; and as regards the third it fixes the direction completely.

In general the supporting forces of a solid structure, whatever their number be, are determined in *six* elements by the condition of vector and locor balance.

Solubility of Joints.—In a plane frame the stress-diagram

for any particular joint-pin cannot be drawn out until all the forces except *two* acting on it have been found.

In a solid structure the stress-diagram for any particular joint-pin can be drawn as soon as all the forces acting on it except *three* have been found.

Flat and solid frames compared

Thus in the plane problems the stress-diagram must be commenced at a joint where only two links meet, unless one of the special methods already explained be employed. But in a solid structure there are no such joints. The stress-diagram is begun at a joint where three links meet, of which joints there are generally two in a stiff, non-redundant, three-dimensional linkage.

These fundamental laws of solid linkages will have their truth made evident in the course of the following illustration, showing how the stress-diagram may be obtained.

Tetrahedral frame.

3. Fig. 93 shows the simplest possible solid linkage—namely, a tetrahedron. It has four joints and six links, thus fulfilling the condition of stiffness $l = 3j - 6$, or $6 = 3 \times 4 - 6$. A B C and A′ B′ C′ are two projections of the structure on two planes at right angles to each other, the dimensions of either of which projections parallel to line G G may be derived from the other by projection perpendicular to G G. The lower view A B C will be called the plan, the upper the elevation, and the line G G the ground line. G G is taken as the plan of the vertical plane of projection of the elevation; and G G is also the elevation of the horizontal plane of projection of the plan.

Lettering. Cyclic order

It is impossible to carry out completely the system of cyclic nomenclature found so convenient in plane diagrams—namely, that of lettering the surface-pens lying between the link-lines. The corresponding system would be to give a letter-name to each *volume*, or *solid pen*, and if this were practicable it would give a perfectly complete and consistent nomenclature for all parts of the frame and stress-diagrams. In the frame of Fig. 93 there is *one* solid inside closed pen, and there are *four* solid outside unclosed pens—namely, one corresponding to

each face of the tetrahedron. At each joint, if the four solid pens meeting at it be taken in a certain cyclic order, the link-line common to any two successive solid pens may be called by the two letters which are the names of these pens; and in this case the two corresponding letters in the stress-diagram would be found at the two ends of the stress-line for this link. But as the drawing is on a flat sheet it is impossible to place the letters on the paper in such a way that they unambiguously indicate the volumes which they are intended to name. Moreover, the choice of the correct cyclic order is difficult and confusing.

Tetrahedral frame

Therefore, the ordinary system of naming the pens of flat structures is here applied to *one* of the projections, and the forces arranged in the proper cyclic order for this *one* view, the lettering of which serves as a complete guide throughout in building up and in reading the stress-diagram. In Fig. 93 it is the *plan* that is thus lettered. The names A B C are given to the plans of the three upper faces of the tetrahedron. Notice that these letters indicate not the faces themselves, but the *plans* of the faces. 1 2 3 4 are the names given to the *plans* of the outside spaces bounded by the plans of the frame and of the outside load-lines. No name is given to the lower face of the frame. This plan now gives a name to each inside and outside link, the left-handed cyclic order being preserved in the particular example now under consideration. The two letters of this name for each link are now written along the elevation of the same link, the letters, however, being accented in order to distinguish readily plan from elevation. This accentuation, although it is here employed in this first example for the sake of clear description, will be found (*vide* subsequent examples) to be unnecessary in practice when the operator has become familiar with the method of procedure.

Lettering. Cyclic order

It is to be noticed that, although the outside force 1 2, 1′ 2′ acts at an outside joint, still in the plan this joint is inside; and, therefore, the pen B in the plan has to be split

in two, B_1 and B_2, and an imaginary link $B_1 B_2$ and joint $(1\ 2\ B_2\ B_1)$ introduced.

The data are the dimensions of the structure; the point of application, the direction, and the magnitude of the load 1 2, 1′ 2′; the direction of supporting force 3 4, 3′ 4′; and the direction of the elevation of the supporting force 2′ 3′. **Tetrahedral frame**

At each soluble joint in a solid frame there act a number of known forces that can be compounded into one known resultant, and three unknown forces acting along the lines of three links whose stresses have not yet been determined. The top joint of Fig. 93 at which the known force 1 2, 1′ 2′ acts is, therefore, typical of the general problem. **Typical Joint**

4. The mode of solution is to resolve the known force into two components—one perpendicular to the plane of *two* of the unknown forces, and the other parallel to this plane. The third unknown force being imagined similarly resolved into two components perpendicular and parallel to the same plane, it is clear that its perpendicular component must be equal and opposite to the perpendicular component of the known load. Because the resultant of the first two unknown forces must be in the plane of these two, and therefore this resultant cannot help to balance the perpendicular component of the known load. **General method**

Thus if from the joint in question along the line of the known force a length be plotted off representing to a convenient scale that force-magnitude, and if from the end of this line be drawn another line parallel to *one* of the three ‘unsolved’ links; and if the intersection of this last line with the plane of the other two unsolved links be determined; then the length of this last line down to its intersection with the plane mentioned measures the stress along the parallel link. The scale, of course, is the same as that to which the known force has been plotted. This second force being thus determined, the remaining two forces can be found at once by closing the polygon by lines parallel to the links.

5. In Fig. 93 the force 1 2 is plotted to scale from the joint to the point *f f′*, and from this point is drawn a line parallel to the link A B to intersect the plane of A C, B C. The plans and elevations of the lines A C, B C are produced to cut the ground line G G. The intersections with G G of the plans A C, B C give the plans of the points where these lines cut the vertical plane; the elevations of these points are found directly underneath their plans on the elevations A′ C′, B′ C′ of the lines; and these elevations being joined, the joining line is the vertical trace of the plane of A C, B C—that is, the line in which this plane intersects the vertical plane. It is marked $V_{AC, BC}$. The horizontal trace $H_{AC, BC}$ is similarly found by taking on the lines A C, B C the plans of the points where the lines A′ C′ and B′ C′ cut G G and joining these plan-points. Only one of these latter need be found, however, because the vertical and horizontal traces necessarily intersect in G G; and thus $H_{AC, BC}$ can be found by joining one of them with the intersection, already found, of $V_{AC, BC}$ and G G. The finding of both points, however, involves very little extra work, and supplies a most useful and much-needed check on the accuracy of the work.

Typical Solid Joint

Construction of diagram

Through *f* and *f′* are now drawn lines parallel to A B and A′ B′. These are plan and elevation of the line through point *f f′* parallel to A B, A′ B′. The plan *f m n* is also the plan of the line of intersection of a vertical plane through this same line and the plane A C, B C.

The plan cuts $H_{AC, BC}$ in a point *m* whose elevation is *m′*, which point is a point in the above-mentioned line of intersection of plane A C, B C and the vertical plane passing through the line through *f* parallel to link A B; it is the point where this intersecting line cuts the horizontal plane. This same intersecting line meets the vertical plane in a point whose plan is *n* where the line ∥ A B through *f* meets G G, this point having the elevation *n′* on $V_{AC, BC}$. Thus *m′ n′* is the elevation of this intersecting line, while *f m n* is its horizontal projection.

The line ∥ A B, A′ B′ through $f f$ lies in this vertica plane, whose plan is $f n$, and it will therefore meet the plane A C, B C in a point in the line in which this latter plane intersects the former. Thus $n' m'$ meets the line $f' q'$ through f' ∥ A′ B′ in the intersection sought for—namely, q'. q' is now projected to q on $f m$. The projections of the stress on bar A B, A′ B′ are $q' f'$ and $q f$. Then from q is drawn a line ∥ A C to meet the prolongation of B C; and from q' is drawn a line ∥ A′ C′ to meet B′ C′ produced. These give the projections of the stresses on the two bars A C, A′ C′ and B C, B′ C′.

But it is better to remove this last part of the construction to another part of the paper, where the stress-diagram can be built up in plan and elevation free from confusion and without each stress-line being repeated. Thus in the plan stress-diagram $b_1 b_2$ ∥ $B_1 B_2$ represents to scale the force 1 2 in horizontal projection, and to same scale $b_1' b_2'$ represents its elevation. When q' is obtained by the preceding construction, there is plotted $b_1' a' \# f' q'$. Then there is drawn $b_1 a$ ∥ B A, and the point a is obtained by vertical projection from a'. The two stress-quadrilaterals are now completed by drawing

Typical Solid Joint

Construction of diagram

$$\left.\begin{matrix} b_2 \\ a \end{matrix}\right> c \ \left\| \begin{matrix} \text{B C} \\ \text{A C} \end{matrix}\right. \text{ in plan and}$$

$$\left.\begin{matrix} b_2' \\ a' \end{matrix}\right> c' \ \left\| \begin{matrix} \text{B}' \text{C}' \\ \text{A}' \text{C}' \end{matrix}\right. \text{ in elevation.}$$

Next in the plan from the joint where 3 4 acts is plotted off along A C to p a length $= a c$. Through the point p so found, a line ∥ 3 4 is drawn, and through its vertical projection p' on A′ B′ is drawn a line ∥ 3′ 4′. The elevation of the intersection of this line with plane A 4 and C 3 is next found. This involves the finding of the traces of this plane; they are found as previously explained for the other plane and are marked $H_{A4, C}$ (horizontal trace) and $V_{A4, C3}$ (vertical trace). The intersection is marked r' in elevation and r in plan. Then $r' p'$ and $r p$ are the projections of the outside force 3 4.

From r is drawn a line parallel to A 4 to meet C 3, and these give the other two stresses acting on this joint in plan. These are inserted in the plan stress-diagram in their proper cyclic order; and from them is deduced the vertical projections, the lines in the vertical stress-diagram being drawn parallel to the elevations of the corresponding links.

Tetrahedral frame

There is next taken the joint at which force 2′ 3′ acts. Here the stresses $3\,c$, $3'\,c'$ and $c\,b_2$, $c'\,b_2'$ are already known. The plane containing their resultant and the link $B_2\,2$ must also contain the outside force 2 3, 2′ 3′. This outside force is, therefore, already partly determined, but the direction in this fixed plane is at the free choice of the designer. In this problem this element of choice is represented by the direction of *one* projection—namely, the vertical 2′ 3′—being included in the data. This being so, the stress quadrilateral for the joint can be immediately completed in elevation; thus $\genfrac{}{}{0pt}{}{3'}{b_2'}\!\!>2'$, the line $b_2'\,2' \parallel (1'\,2')\,B'$. This gives 2′ in elevation; its plan is next found by projection downwards from 2′ to 2 on line drawn through $b_2 \parallel B_2\,2$. The point 2 so found is now joined to 3, and 2 3 gives the plan of this force in direction and magnitude.

Construction of diagram

Next the plan and elevation parallelograms for the imaginary joint $1\,2\,B_2\,B_1$ are drawn in; and finally the points 4, 1, and 4′, 1′ are joined. These latter give the direction and magnitude in plan and elevation of the remaining supporting force.

Plan and elevation Stresses

6. This completes the drawing of the stress-diagram. The stresses are known by their plans and elevations. Their unresolved magnitudes are to be found by compounding the plan with the vertical component measured from the elevation. A convenient mode of effecting this composition graphically for all the stresses in one 'composition diagram' is shown in Fig. 95 (*see* Section 8).

7. When the stresses in a large and complicated structure

have to be investigated, generally the known loads are more or less numerous. The frame is supported by at least three forces acting at known points, and the first part of the problem is to calculate these supporting forces. Of these the condition of equilibrium is sufficient to determine *six* elements, as in the last problem, where the three magnitudes, the plan and elevation directions of one force, and the plan direction of another force are calculated. To find these six elements it is necessary to reduce the known loads to as simple a resultant as possible; but, as has been explained in Chapter VIII., Section 8, they cannot generally be reduced to a *single* resultant force. Their simplest equivalent is a pair of non-intersecting forces at right angles to each other. The mode of reducing them to this simplest form is fully given in Chapter VIII., Section 8, Fig. 58 *a*.

General case of three supports

In Fig. 94, in the three orthogonal rectangular projections (π) (ε) and (σ), the three resultants of the projections of the known forces lie along the lines $\rho_\pi\, \rho_\epsilon\, \rho_\sigma$. These have been found by adding together the several projections in three vector-diagrams so as to obtain the vector sum in each projection, choosing for each a suitable pole, and from the three pencils radiating from these poles drawing three single pens.

Pentahedral construction

The three points $\alpha\, \beta\, \gamma$ are known as points in the lines of action of the three supporting forces 51, 34 and 45. In elevation these three are all given at the same level and the ground line O E is taken through them. It is taken so because in most engineering problems it is easy to make one of the planes of projection (usually the plan) pass through the three points of support.

The plan directions of these three supporting forces are also among the data. The six elements to be calculated from the condition of equilibrium are in this example the three magnitudes and the three directions in elevation.

In this case the plans of the magnitudes can be found at once. In (π) produce ρ_π or 12 to meet 34 in μ, and produce

45 and 51 to meet in ν. Join $\mu\nu$. The resultant $(\rho_\pi + 34)$ and also the resultant $(45 + 51)$ must each lie along the line $\mu\nu$, and be equal and opposite. In a convenient position on the part of the paper where the *plan stress-diagram* is to be drawn, plot $d_1 d_2 \#\!\!= \rho_\pi$, or simply take it as already obtained in the previous part of the construction and mark it $d_1 d_2$. Then draw $d_2 4' \parallel 34$ and $d_1 4' \parallel \nu\mu$, and mark the intersection $4'$. Then draw $4' 5' \parallel 45$ and $d_1 5' \parallel 51$, marking their joint $5'$. Then $d_2 4'$, $4' 5'$, and $5' d_1$ are the three plan magnitudes sought.

General case of three supports

Next take any point δ' in (ϵ) on ρ_ϵ and project it horizontally to δ'' on ρ_σ in (σ). Measure the horizontal distance of δ'' from O V and plot it vertically downwards in (π) to δ from line O E on the vertical projection line through δ'. In the elevation stress-diagram d_1 and d_2 are vertically over $d_1 d_2$ in (π), and in (ϵ) $d_1 d_2$ is horizontal. $d_2 e'$ is vertical and $\#\!\!= \rho_v$, the vertical component of ρ; and $d_1 e' \#\!\!= \rho_\epsilon$. The whole set of known forces has now its equivalent in (π) as ρ along line 12 and at level δ', and ρ_v through δ.

Pentahedral construction

Next draw $\delta\lambda \parallel \rho_\pi$ in (π) to meet in λ the line $\alpha\gamma$, and project λ upwards to λ' on ρ_ϵ in (ϵ). The same set of forces is also equivalent to ρ_π acting horizontally in the line 12 at the level of λ' and ρ_v acting vertically through λ.

Next mark θ in (π) where ρ_π meets $\alpha\gamma$, and project θ upwards to θ' in (ϵ) on the horizontal line through λ'.

Now suppose erected a 9-bar linkage, or pentahedron, on the base $\alpha\beta\gamma$ with its two other joints at $(\lambda\lambda')$ and $(\theta\theta')$. At joint $(\lambda\lambda')$ suppose ρ_v to act, and the force ρ_π to act at joint $(\theta\theta')$. Draw in the bars of this linkage in (π) and (ϵ) and letter the spaces in plan as shown. The right-hand cyclic order has been chosen. In plan the bars D E and E (1 2 3) lie coincidently with the bars E A, D B, and D C. In order to obtain places for the letters E and D the former two bars are represented by curved dotted lines; but, of course, in order to draw lines parallel to these bars in the plan stress-diagram, the directions are taken from the straight line $\alpha\gamma$. It is

especially important to have these spaces clearly marked, because D is divided by the external force line 1 2 into two, viz. D_1 and D_2; and E is divided into three, E_1, E_2, E_3, by the external forces 1 2 and 2 3. This last 23 is ρ_v, which is really vertical and represented in the plan stress-diagram by a point only, but in (π) it is represented by a finite line for the same reason that curved lines are drawn to represent the above-mentioned bars. Note that the bar D B between the two joints $\lambda\, \theta$ is horizontal.

The linkage is taken of this shape with the faces $\alpha \lambda \gamma$ and $\lambda \theta \gamma$ in the same vertical plane because so doing very much simplifies the resolution of the forces at joints λ and θ.

Taking first joint θ, the horizontal component of stress in B C normal to $\alpha \gamma$ must balance that of $\rho_\pi \# d_1 d_2$ in the same direction. Therefore draw $d_2 c' \parallel$ B C and $d_1 c' \parallel \alpha \gamma$. This gives $d_2 c'$ the plan of stress in B C. In (ε) draw $d_2 c' \parallel$ B C and obtain c' by vertical upward projection from c' in (π). Next in (ε) draw $c' c \parallel$ D B and $d_1 c \parallel$ D C, and then $c b \parallel$ C B to meet $d_1 d_2$ in b. These points, c and b in (ε), give $d_2 b$, $b c$, and $c d_1$ the (ε) projections of the stresses in D B, B C, and C D. Next in (π) draw $d_2 b$ and $d_1 c$, both $\parallel \alpha \gamma$, and on these obtain b and c by projection downwards from b and c in (ε). Check the accuracy of drawing by finding $b c \parallel d_2 c' \parallel$ B C. **Pentahedral Construction**

Take next joint λ. The force $E_2 E_3$ and all the others, except that along A B, lie in one vertical plane, and the plan-projection of their resultant must therefore lie along line $\alpha \gamma$. Therefore, there is zero stress on A B, and a must be marked along with b in the stress-diagram both in (π) and in (ε). The known forces acting at this joint are $a d_2$ and $d_2 e'$ in (ε). Draw $a e_3 \parallel$ A E and $e' e_3 \parallel$ D E, and mark the joint of these e_3. Draw $d_2 e_2 \# e' e_3$ and $e_3 e_2 \# e' d_2$. Next in (π) draw $d_2 e_2 \parallel \alpha \gamma$, and find on it e_2 and e_3 (coinciding) by projection from $e_2 e_3$ in (ε). Next find e_1 in (π) and (ε) by completing the parallelograms $e_2 d_2 d_1 e_1$.

For the joint at α in (π) draw $4' 4 \parallel \alpha \gamma$ and $a 4 \parallel$ A 4, meet-

ing in 4. Then draw $4\,3 \,\#\, 4'\,d_2$, the point 3 falling in line $e_3\,a$ produced. The plan of the stress-diagram for this joint is now complete, viz. $a\,e_3\,3\,4\,a$. Complete it in (ε) by drawing $e_3\,3 \parallel E_3 3$, i.e. horizontal, and finding in this line the point 3 by projecting upwards from 3 in (π); also finding 4 in the horizontal line through a by projecting upwards from 4 in (π); and finally joining 3 4.

Pentahedral Construction

Next for joint β there are already obtained in (π) the lines $c\,b\,a\,4$. Draw $4\,5 \,\#\, 4'\,5'$, and join $5\,c$. The accuracy of the drawing is checked by finding $5\,c \parallel 5\,C$. In (ε) draw $c\,5 \parallel C\,5$, i.e. horizontal, and find on this 5 by projection from 5 in (π). Also join 4 5 in (ε). The diagram for this joint in (ε) is $c\,b\,a\,4\,5\,c$.

Next, in both (π) and (ε) complete the parallelogram $3\,e_3\,e_2\,2$, giving 2 coincident with 3 in (π) and making $2\,3 \,\#\, \rho_v$ in (ε); and also the parallelogram $2\,e_2\,e_1\,1$, which in (ε) gives 1 on the horizontal line $e_1\,e_2$, and makes $1\,2 \,\#\, e_1\,e_2 \,\#\, d_1\,d_2$. Join 5 1 in both ($\varepsilon$) and ($\pi$), and check by finding in (π) $5\,1 \parallel 5'\,d_1$.

The diagram for joint γ will now be found complete in both (π) and (ε), namely, $5\,1\,e_1\,d_1\,c\,5$.

Composition Diagram

The elevations of the three supporting forces have now been found, as well as their plans, and their resultant magnitudes are found on the auxiliary 'composition diagram' by plotting from O′ horizontally and vertically the plans and the vertical components of the elevations. On this diagram is also seen the resultant magnitude of the datum-load, viz. 1 3.

Shorthand symbols

8. In these constructions, in which the stresses are shown by elevation and plan, a point is very frequently to be found partly by help of projection from elevation to plan, or *vice versâ*. In order to write out the construction for a complicated frame in a form that is not excessively tedious, it is advisable to adopt some shorthand symbol to represent this operation. In what follows the symbols used are ↑ ↓ ⤉ and ⤈. The phrase '(π) d ⤉ e' means 'find the point e in plan (π) by drawing from d, already found in plan, a line parallel to D E in plan, and projecting vertically downwards on to this line

from the point e already found in elevation.' Again '$(\varepsilon)\ g \ddagger f$' means 'find f in (ε) by drawing from g in (ε) parallel to G F in (ε) and projecting upwards from f in (π).' Not infrequently the diagram for a joint cannot in the first place be drawn in correct cyclical order; it has to be otherwise drawn at first, and after finding the closing lines the polygon has to be partially redrawn to put it into proper cyclic order. In that case temporary lines have to be drawn in the stress-diagram parallel to frame-lines named differently. To express this clearly the foregoing symbols are, when needful, expanded thus: '$(\pi)\ d \ddagger e \parallel$ G E' means 'in (π) draw from d a line $\parallel$ G E and find on it the point e by downward projection from e in (ε).' Also '$\begin{matrix} c \\ f \end{matrix}\!\!> k \left\| \begin{matrix} \mathrm{D\,E} \\ \mathrm{L\,M} \end{matrix} \right.$' means 'draw from c and f two lines parallel to D E and L M, and mark their intersection k.' Again, '$p\ q = 0$' means 'mark q coincident with already found point p.' **Short-hand symbols**

9. In Fig. 95 is shown a more complicated solid structure. It represents a braced pier for a bridge, although, for the sake of greater clearness in the illustration, proportions have been chosen that deviate somewhat from those commonly found in such piers. The three views, (π), (ε), and (σ), are plan, front, and side elevations. The latter is used only in finding the projection ρ_σ of the resultant of the known loads, for the sake of determining the unknown supporting forces. In each projection those bars which do not lie on the nearer surfaces of the framework are drawn in long-dash lines, as if the bars were hidden from view. Although they are actually seen between the front bars, this makes the diagram less liable to confusion. A horizontal bracing diagonal crosses the rectangular base, and another horizontal diagonal crosses at the level of the second tier of joints. An examination of the relation between the number of joints and of links will show that the structure is stiff but non-redundant, there being sixteen joints and forty-two links. **Braced Pier**

In one respect the graphic solution of this framework is easier than the general case shown in Fig. 93. This results from all the bars except two lying in four main planes, which are, taken in pairs, perpendicular to two planes of easy projection. Thus two of these planes are represented by single lines in the front elevation (ε). Again, the plane containing the horizontal bars meeting at any joint is also represented by a single line in (ε). This very much simplifies the part of the construction equivalent to finding the components perpendicular to these planes; this perpendicular being in (ε) normal to the line-projection of the plane. Otherwise expressed, these line-projections are the vertical traces of the planes, and thus the tedious operation of finding the traces of a large number of planes is obviated; this operation having, in fact, in this example to be performed once only. As the majority of engineering structures are of this character, the present problem is a fair illustration of the ease or difficulty experienced in applying the method in practice.

Simplification

The pier is secured to the foundation at the bases of the four main columns, which rake inwards towards the vertical axis of symmetry. One, viz. 7 8, of the four forces exerted by the foundation is supposed to be known both in magnitude and direction. The plan-directions of the other three are also given. Their magnitudes and directions in elevation are found by construction. This construction is shown in the figure by close dotted lines, but it is not here described, as it is precisely similar to that last given in Fig. 94.

Braced Pier

At the four top joints are applied vertical loads, each equal to 28 tons. At six of the joints on one side act equal wind-forces, the elevation and plan-directions of which are given. The elevation-projection of each measures 12 tons, and its plan-projection about 21 tons, the whole resultant magnitude of each being 21·7 tons. In the plan, the pens of which are lettered, these wind-force lines cut through the bars so as to cut up the pens very much and necessitate

the introduction of many imaginary joints. The right-hand cyclic order is adopted, taking it from the lettering of the plan. The central pen in plan is named A. This is split up into four pens, A_1, A_2, A_3, A_4, in order to afford distinct names for the four vertical loads. The plan of each of these is a point, but their plans are marked by short dotted lines for the sake of showing a boundary line between spaces A_2 and A_3, &c. Of course, in the plan stress-diagram a single point (marked a) represents all these four loads. **Braced Pier**

These loads and the wind-forces are first plotted in plan and elevation, the wind-forces *succeeding* the vertical loads. The same set of forces is next plotted in the side elevation (σ), the magnitudes and directions being obtained from the components shown in the two other projections. Three poles are chosen: from them three single-pens are constructed; and by this means the three resultants, ρ_π, ρ_ϵ, and ρ_σ, are found. Then the construction of Fig. 94 is applied to find the supporting forces, 8 9, 9 10, and 10 1, which are plotted in (ε) and (π), forming in each a closed polygon along with the previously known loads and 7 8. **Loads and Wind Forces**

The first joint taken is (A I B). Here the two links A I and B A are horizontal, and, therefore, the vertical component of the thrust of the strut I B must balance the load $a_4\,a_1$. Also B A is normal to plane of (ε). Therefore in (ε) mark b coincident with a_4 and draw $\genfrac{}{}{0pt}{}{b}{a_1{}'}\!\!>i'$. Next in ($\pi$) draw $a \ddagger i$ and $\genfrac{}{}{0pt}{}{a}{i}\!\!>b$. With the help of the nomenclature already mentioned, the construction (including these first steps) is described as follows. The points $i\,h\,p\,q\,y\,a_1$ 1 in (ε) are simple repetitions of $i'\,h'\,p'\,q'\,y'\,a_1{}'$ 1′. **Stress-Diagram**

Joint (A I B). $\quad(\varepsilon)\ a_4\,b = 0 \quad \genfrac{}{}{0pt}{}{b}{a_1{}'}\!\!>i'.$

$$(\pi)\ a \ddagger i \quad \genfrac{}{}{0pt}{}{a}{i}\!\!>b.$$

Joint (A E F). $(\varepsilon)\ a_2 f = 0 \quad f e_4' \mathrel{\#} 4\,5 \quad {}^{e_4'}_{a_3}\!\!>e_4 \parallel \begin{smallmatrix}\text{F E}\\ \text{A E}\end{smallmatrix} \quad {}^{f}_{e_4}\!\!>e_5.$

$(\pi)\ a \ddagger e_4 \quad e_4 e_5 \mathrel{\#} 4\,5 \quad {}^{e_5}_{a}\!\!>f.$

Joint (A F G H I). $(\varepsilon)\ a_1 i \mathrel{\#} a_1' i' \quad {}^{f}_{i}\!\!>h \parallel \begin{smallmatrix}\text{F G or G H}\\ \text{I H}\end{smallmatrix}.$

Owing to symmetry of construction $i h = 0$.

$(\pi)\ i h = 0 \quad {}^{h}_{f}\!\!>g \qquad (\varepsilon)\ h \ddagger g.$

Braced Pier

Joint (A B C D E). $(\varepsilon)\ e_4 d'_3 \mathrel{\#} 3\,4 \quad {}^{d_3'}_{b}\!\!>d_3 \parallel \begin{smallmatrix}\text{E D}\\ \text{B C or C D}\end{smallmatrix}$

$d_3 d_4 \mathrel{\#} 3\,4$. Join $d_3\, e_4$.

$(\pi)\ e_4 \ddagger d_4 \quad d_4 d_3 \mathrel{\#} 3\,4 \quad {}^{d_3}_{b}\!\!>c. \qquad (\varepsilon)\ b \ddagger c.$

Joint (H G O P). Both G O and P H are horizontal; ∴ vertical component of $h\,g =$ vertical component of $o\,p$. But O P is in same line with H G; ∴ $op = g\,h$.

$(\varepsilon)\ g\,o = 0 \quad h\,p = 0. \qquad (\pi)\ g\,o = 0 \quad h\,p = 0.$

Joint (D C K L). $(\varepsilon)\ c\,k = 0 \quad d_3 l_2' \mathrel{\#} 2\,3 \quad {}^{l_2'}_{k}\!\!>l_2 \parallel \begin{smallmatrix}\text{D L}\\ \text{K L}\end{smallmatrix}$

$l_2 l_3 \mathrel{\#} 2\,3$. Join $l_3\, d_3$.

$(\pi)\ d_3 \ddagger l_3 \quad l_3 l_2 \mathrel{\#} 2\,3 \quad {}^{l_2}_{c}\!\!>k.$

Plan and Elevation Stress-Diagram

Joint (K C B I H P Q J). $(\varepsilon)\ {}^{k}_{p'}\!\!>q' \parallel \begin{smallmatrix}\text{K J or J Q}\\ \text{P Q}\end{smallmatrix}$. This gives $p'\,q' = 0$.

$(\pi)\ p\,q = 0 \quad {}^{q}_{k}\!\!>j. \qquad (\varepsilon)\ k \ddagger j.$

Joint (J Q Y R). Y R is in line with J Q; ∴ $qj = y\,r$ and $q\,y = 0 = r\,j$.

$(\varepsilon)\ q'\,y' = 0 \quad j\,r = 0 \quad (\pi)\ q\,y = 0 \quad j\,r = 0.$

Joint (O G F E D L M N).

$(\varepsilon)\ d_4 d_5 \mathrel{\#} 4\,5$; join $e_5\, d_5 \quad l_3 l_4 \mathrel{\#} 3\,4 \quad l_4 l_5 \mathrel{\#} 4\,5$;

join $d_5\, l_5 \quad l_5 m'_6 \mathrel{\#} 5\,6 \quad {}^{m'_6}_{o}\!\!>m_6 \parallel \begin{smallmatrix}\text{M L}\\ \text{O N or N M}\end{smallmatrix}$

$m_6 m_5 \mathrel{\#} 5\,6$; join $m_5\, l_5$.

$(\pi)\ d_4 d_5 \# 4\,5 \quad d_5 \ddagger l_5 \quad l_5 \ddagger m_5 \quad m_5 m_6 \# 5\,6$

${}^{m_6}_{o}\!>n \qquad (\varepsilon)\ o \dagger n.$

Joint (N M U V). $(\varepsilon)\ n\,v = 0 \quad m_6 u'_6 \# 6\,7 \quad {}^{u'_6}_{v}\!>u_7 \,\Big\|\, \begin{matrix}\mathrm{MU}\\ \mathrm{VU}\end{matrix}$

$u_7\,u_6 \# 6\,7$; join $u_6\,m_6$.

$(\pi)\ m_6 \ddagger u_6 \quad u_6 u_7 \# 6\,7 \quad {}^{u_7}_{n}\!>v.$

Joint (8, 9 X W). $(\varepsilon)\ 8\,w = 0 \quad {}^{9}_{w}\!>x$

$(\pi)\ 9 \ddagger x \quad {}^{x}_{8}\!>w.$

Joint (10, 1 T S). $(\varepsilon)\ 10\,s = 0 \quad {}^{1}_{s}\!>t_1$ **Braced Pier**

$(\pi)\ 1 \ddagger t_1 \quad {}^{t_1}_{10}\!>s.$

Joint (7 8 W V U T).

At this joint the horizontal diagonal bracing bar will be called $V_1\,V_2$. At its other end it will be called $R_1\,R_2$. In the stress-diagram its stress will be repeated and will appear as $v_1\,v_2$ and $r_1\,r_2$, the two equal and opposite pulls this bar exerts on the two joint-pins at its ends. At this joint the three unknown forces are $w\,v, v_1\,v_2$, and $u\,t$, the thrust $t_7\,7 = t_1\,1$ being already found. These three are the stresses along the bars W V, $V_1\,V_2$, and U T. Find the traces in (π) and (ε) of the plane containing W V and $V_1\,V_2$. They are marked $(\mathrm{w\,v},\ \mathrm{v_1\,v_2})_\pi$ and $(\mathrm{w\,v},\ \mathrm{v_1\,v_2})_\varepsilon$ in the diagrams, and are drawn in dash-dot (—·—·—) lines. In both (π) and (ε) stress-diagrams plot $7\,t_7 \# 1\,t_1$. Also in both (π) and (ε) plot from w, the known stress $v_1\,u_7$ along the bar V U; thus $(\pi)\ w\,v' \# v\,u_7$ and $(\varepsilon)\ w\,v' \# v\,u_7$. This gives $t_7\,v'$ the resultant of the known forces acting at the joint in both (π) and (ε). Plot this backwards from the joint in both plan and elevation of the pier to the point marked α. In (π) from α draw $\alpha\,\beta\,\gamma \parallel$ U T, meeting $(\mathrm{w\,v},\ \mathrm{v_1\,v_2})_\pi$ in β and the ground-line in γ. Project β to β' on the ground-line, and γ to γ' on $(\mathrm{w\,v},\ \mathrm{v_1\,v_2})_\varepsilon$. Draw $\gamma'\,\beta'$

Plan and Elevation Stress-Diagrams

to meet in η' the line $\alpha' \eta' \parallel$ U T in (ε). Project η' downwards to η on $\alpha \beta$. Then $\eta' \alpha'$ and $\eta \alpha$ are the (ε) and (π) projections of the stress on U T. Then proceed thus :—

$$(\pi)\ t_7 u_7' \#\ \alpha \eta \quad u'_7 v_2 \#\ w v' \quad {}^{v_2}_{w}\!\!>v_1 \ \Big\| \begin{matrix} V_1 V_2 \\ W V \end{matrix}$$

$$(\varepsilon)\ t_7 u_7' \#\ \alpha' \eta' \quad u'_7 v_2 \#\ w v' \quad {}^{v_2}_{w}\!\!>v_1 \ \Big\| \begin{matrix} V_1 V_2 \\ W V \end{matrix}.$$

Joint (R Y X 9, 10 S).

Braced Pier

$$(\pi)\ s s' \#\ v_2 v_1 \quad s' r' \#\ r y \quad {}^{r'}_{x}\!\!>y_2 \ \Big\| \begin{matrix} S R \\ X Y \end{matrix}$$

$$y_2 r_2 \#\ y r \quad r_2 r_1 \#\ v_1 v_2. \quad \text{Join } s r_1.$$

This gives $s r_1 \#\ r' y_2$. The force-balance diagram in (π) for this joint is now x 9, 10 $s r_1 r_2 y_2 x$, the last line $y_2 x$ being the stress on bar Y X.

$$(\varepsilon)\ s \updownarrow r_1 \parallel S R \quad r_1 \updownarrow r_2 \parallel R_1 R_2 \parallel V_1 V_2, \text{ or } r_1 r_2 \#\ v_2 v_1$$

$$r_2 \updownarrow y_2 \text{ or } r_2 y_2 \#\ r y'. \quad \text{Join } y_2 x.$$

Check by finding $y_2 x \parallel$ Y X. This furnishes the first important check on the accuracy of the drawing, although throughout the previous construction many subsidiary checks have been afforded by reason of a point being able to be derived in either of two ways, viz. either by projection to (ε) from (π) or *vice versâ*, or else by drawing parallel to a line in same view as the point to be found.

Joint (V W X Y Q P O N).

Plan and Elevation Stress-Diagrams

All the forces acting at this joint have already been found except that exerted by the horizontal bracing diagonal; and a second check is afforded here by finding that the resultant of these already determined forces is parallel to this bar. The summation of the forces is not yet shown on the diagram; in order to get them added together consecutively it is necessary to repeat some of them. Thus, beginning with the force $v w$, there is already plotted consecutively $v_1 w x y_2$ in both (π) and (ε). Next mark q_2 and p_2 coincident with y_2 in both (π) and (ε), since $y q = 0 = q p$.

Then in both plot $p_2 o_2 \# p o \quad v_1 n_1 \# v n \quad n_1 o_1 \# n o$.

Join $o_1 o_2$ = the stress on the horizontal diagonal. This should be found parallel to this bar $O_1 O_2$ in *both* (π) and (ε). This check is therefore a *double* one.

Joint (K J R S T U M L). All the stresses on the bars acting at this joint have already been found; and, although they are not yet built consecutively into a polygon, this can be now done. If the drawing is accurate the polygon will close in both (π) and (ε). The double check obtained here is not really independent of that given at the previous joint; if this previous check showed exact accuracy the latter one should also do so except for some *new errors* occurring in the necessary transference of the lines for the making of this last polygon. The polygon is **Braced Pier**

$$k_1 j_1 r_1 s t_1 t_2 u_2 m_2 l_2 k_2 k_1.$$

From the plan and elevation stress-diagrams is constructed the 'composition stress-diagram' in the manner already described. On it the whole stress on each bar in the structure can be read off to scale along a diagonal line (not drawn) between two similarly named points on the vertical and horizontal axes. **Composition Diagram**

10. Although at one stage of the construction of the last figure it was found impossible to proceed *directly* from the part of the structure already solved to a *contiguous* soluble joint, this difficulty was surmounted by next taking two isolated soluble joints and from them working up to the point at which the difficulty occurred. This sort of difficulty occurs occasionally in solid structures, just as it does in plane ones, and it cannot always be evaded in the manner of the last example. For this reason recourse must sometimes be had to the 'method of solid sections.' The complete section of the structure must cut six bars only whose stresses have not already been determined. The whole external load applied to the part of the structure lying to either side of the section **Method of solid sections**

is known, and thus the problem is to find the six magnitudes of six forces, whose lines of action are known, such as will balance a known load.

The following method supposes that the six given bar-lines can be taken in three pairs, each of which pairs intersects. This is not an hypothesis invariably true, but it will cover most cases met in engineering practice. Let the three pairs intersect in the three points $\alpha\,\beta\,\gamma$. Call the plane of the pair meeting in α by the name A; that of the pair meeting in β by B; and that of those meeting in γ by C. Let all the known forces acting on the structure on one side of the section be compounded to a pair of forces, one, say ρ_π, lying in the plane $\alpha\,\beta\,\gamma$; the other, say ρ_v, perpendicular to this plane. The latter cuts the plane in a certain point whose distances from the three lines $\alpha\,\beta$, $\beta\,\gamma$, and $\gamma\,\alpha$ can be measured.

Method of solid sections

Imagine now the forces exerted by the two bars in plane A compounded into a single force, say a, acting through α; and this again resolved into two components, one a_π in the plane $\alpha\,\beta\,\gamma$, and the other a_v perpendicular to it. The distance of the latter from line $\beta\,\gamma$ being known, its magnitude can be found because its moment round $\beta\,\gamma$ balances that of ρ_v round the same axis. The two pairs of stress-forces acting at β and γ are similarly treated, and the components at these points perpendicular to plane $\alpha\,\beta\,\gamma$ found by taking moments round $\gamma\,\alpha$ and $\alpha\,\beta$. These moment calculations can each be carried out in the ordinary graphic manner.

The component a_π lying in plane $\alpha\,\beta\,\gamma$ of the stress-force at α can now be resolved into two components, one perpendicular to the line of intersection of plane A and plane $\alpha\,\beta\,\gamma$ and the other along this line. The former has a definite ratio to a_v already found (viz. it equals $a_v \times$ cotangent of angle between planes A and $\alpha\,\beta\,\gamma$), and can be at once obtained by graphic construction. The other acts along a line of known direction and position. The problem is thus reduced to finding the magnitudes of three forces acting along known lines in

one plane $\alpha\,\beta\,\gamma$, such as to balance a known resultant force in this plane. The solution of this problem has already been several times used; for instance, in the last figure. The three components of a are now compounded into a single force, and this once more resolved into two components along the two given lines of the bars lying in plane A; these last being the stresses on these bars. The stresses on the four bars meeting at β and γ are calculated in the same manner.

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